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Arithmetic for Those Who Excel

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THIS TOPIC DEALS with the problem of providing for the superior pupil in the field of arithmetic. Sometimes this pupil is designated as the fast learner, the gifted pupil, or by some other descriptive phrase. The title of this paper makes it comparatively easy to identify the pupils whose achievement in arithmetic is significant. McWilliams and Brown¹ recommended that superior pupils should be considered those in the upper 20 per cent in general intelligence whose abilities lend themselves readily to intellectual pursuits. We shall not attempt to designate a fixed per cent of a class as belonging to the group which excels in arithmetic. Those students who understand the work and develop insight into the subject constitute the group who need some modification of the program which is designed for the average or below average in achievement in this subject. The problem is how to provide enrichment for the pupil who shows superiority in the field of arithmetic.

Purposes of Enrichment

The purposes or objectives of enrichment for the fast learners are as follows:

¹ McWilliams, Earl M. and Brown, Kenneth E. *The Superior Pupil in Junior High School Mathematics*, p. 3. Bulletin 1955, No. 4. U. S. Department of Health, Education, and Welfare, Office of Education.

1. To encourage pupils to work independently and learn directly from appropriate books and periodicals.

2. To enable pupils to form broader skills and to acquire more technical knowledge than the average pupil can assimilate.

3. To give the pupil the opportunity to explore, discover, and develop his interests and potentialities in mathematics

4. To challenge the pupil to work at the level of operation at which optimum growth in arithmetic is possible.

The second and third objectives are self evident, but the first and last aims may need clarification. One of the highest forms of learning consists in working independently from others and mastering a given topic or subject. A prime function of school instruction should be to enable a student to identify and solve problems by himself. The fourth objective states that a student should be challenged to operate at a level at which he will have maximum growth in arithmetic. Good teaching is in evidence when a pupil is challenged to operate at the highest level of abstraction at which he understands the work. If the same kind of challenge offered to pupils of average or below average ability is given to the pupils in the superior group, the work is geared to such a low level of abstraction that the superior pupil is not sufficiently challenged to exhibit maximum growth in arithmetic.

Suggested Ways of Meeting Needs of Superior Pupils

A survey of the literature dealing with the problem of meeting the needs of the superior pupil in arithmetic shows that four methods or procedures are used. They are:

1. Acceleration
2. Segregation
3. Enrichment
4. A combination of any of the first three.

Acceleration means that the pupil will cover the year's work in less than normal time. McWilliams and Brown² stated that many junior high school teachers believe that mathematics is the one subject area where the method of acceleration is the best possible means of providing for the superior pupils.

Segregation implies that special classes are formed for the superior pupils. In New York City certain schools function solely for the gifted pupils.

The third plan calls for enrichment of the curriculum. Some of the activities in a plan of this kind include the following:

- a. Additional exercises and problems, frequently starred because of difficulty
- b. Vocabulary study
- c. Individual and group reports on topics investigated
- d. Finding social applications of mathematics in community resources
- e. Use of supplementary textbooks
- f. Mathematics clubs
- g. Mathematics exhibits and contests
- h. Making of models and other visual aids.

It is apparent that acceleration and segregation are administrative devices for dealing with a vital problem in learning. A problem of this kind cannot be solved by such artificial means.

² *Ibid.*, p. 28.

The activities listed under enrichment touch upon the problem of learning. These activities are favorable, but they are inadequate. The problem of providing adequate enrichment can never be solved, but more adequate means of dealing with it can be offered than the four procedures listed above.

Miss Cortage³ proposed two principles governing accepted diet rules for the scientific feeding of gifted children. They are:

1. When the pupils have cleared their plates, do not fill them again with the same kind of food.
2. If you want pupils to grow, mentally and physically, do not try to shove into them double portions of food per person. Why not offer different foods to them?

The first of these two generalizations may be interpreted in arithmetic to mean that it is unprofitable to provide the fast learner with additional exercises and problems of the same kind so as to keep him occupied. Many teachers of arithmetic have followed this practice which should be condemned. The teacher discovered that superior pupils could complete an exercise in less time than slower learners. To meet the problem of keeping all pupils active, teachers demanded more drill for the pupils who needed it the least. Frequently, this led to the use of a workbook which merely supplied more drill. The teacher could use the plan of supplying more drill exercises and more problems to solve until the bright pupils discovered that it did not pay to be bright. Then these pupils would initiate the slow-down so as to adjust their working time to that of the other pupils in the group.

The second principle states that double doses are to be avoided. Acceleration is a modern name for double doses. Instead of

³ Cortage, Cecelia. "Postscript—To Do or Not to Do," *The Gifted Child in the Elementary School*, p. 132. California Elementary School Administrators' Association: Twenty-sixth Yearbook, 1954.

acceleration, the teacher should have an enriched program which will challenge the bright pupil as he progresses at a normal rate. Therefore, the problem of providing an adequate program for pupils who make high achievements in arithmetic resolves itself into the question of finding challenging problems and activities so that the pupils can proceed at the normal rate in the school program. This plan must be one which is easily administered and can be used by most teachers in most classrooms. It follows, then, that there must be some other method or procedure besides the four enumerated for meeting the needs of those who excel in arithmetic.

Differentiation of Curriculum and Level of Operation

The fifth plan calls for differentiation of both the curriculum and the level of abstraction at which the pupil operates. The proposals for enrichment previously enumerated represent differentiation of the curriculum. Except for the first two items pertaining to starred exercises and problems and vocabulary study, all of the suggested items or activities function without the framework of the textbook. The proposed plan of differentiation of the curriculum and of level of operation should take place within the framework of the textbook.

The textbook in arithmetic is the chief instructional tool in the classroom. Unfortunately, its potentialities as an effective instrument of learning have not been utilized fully. To a large extent the textbook functions as a book designed to provide graded exercises for drill and problems for solution. The same pattern of solution for both examples and problems is considered a desirable goal for the class to achieve. Under such conditions the textbook can never be used as a satisfactory instrument for enriching the program for fast learners in arithmetic.

Let us first look at the structure of a typical heterogeneous class. There are

certain to be some slow learners as well as some fast learners. The achievement of most of the group will fall within a range of what may be termed an average group. In order to meet the range of abilities in a typical class, ability grouping based on achievement and development of insight into the subject would be both essential and desirable. This type of activity is practiced very successfully in reading, but sparingly in arithmetic.

Grouping within the class provides many or most of the advantages of segregation for special classes. The word segregation should not be used because of the unfavorable practices associated with this term. The groups formed should be fluid and not static. There are many times in an arithmetic class period in which the class should function as an entity, or one group. Subgrouping would be unprofitable at these times. To illustrate, a third grade made a study of the meaning of the terms, *pounds* and *ounces*. The class took excursions to places where scales or other weighing devices were used. The pupils weighed objects and discovered the kinds of objects in which the weights should be expressed as pounds, ounces, or both. In an activity of this kind grouping is undesirable.

Let us consider the introduction of common fractions in the fifth grade. Very probably each pupil would have a kit of fractional cut-outs to use to discover the meaning of a fraction and some of the principles which govern the operations of fractions. During the period of exploration the class should function as an entity. The more able pupils would discover more quickly than the slow learners the properties of fractions and how fractions may be combined by addition and subtraction. After the class has discussed how to add or subtract fractions, the teacher should form groups according to the pupil's ability to understand and succeed in dealing with fractions in these operations. The class should be doing the same work from the

textbook, but the two or three groups formed would be working at different levels of abstraction. The slow learners might use the materials of his kit to add such fractions as $\frac{1}{2}$ and $\frac{1}{6}$ or $\frac{3}{4}$ and $\frac{1}{2}$. The fast learners would not use supplementary aids in solving the examples. These pupils would work with symbols. The pupil should be able to estimate a sensible or reasonable answer. Then he should check to see if the answer he got in his solution is sensible. In the example $\frac{1}{2}$ and $\frac{1}{6}$, a pupil's thought might be as follows: " $\frac{1}{6}$ is less than $\frac{1}{4}$. Since $\frac{1}{2}$ and $\frac{1}{4}$ are $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{1}{6}$ must be less than $\frac{3}{4}$."

In a self contained classroom the pupils should function as a class in dealing with certain social applications of number and in the introductory work of building concepts. Ability groups are formed for dealing with different levels of abstraction in operation in presenting a topic or process. The groups are fluid and change as different topics or processes are introduced.

The teacher must be given prime consideration in any plan which is proposed for meeting the problem of individual differences. Frequently, a theoretical plan to meet these differences neglects the role of the teacher in implementing the plan. For a program to be successful, there must be a minimum number of daily lesson plans for the teacher to prepare. If a teacher has three groups within her class proceeding at different rates, or working with different subjects or topics, it is apparent that she will soon have the class functioning as a class and not as three separate groups. If ability grouping is practiced in arithmetic, the same practice could be anticipated in dealing with the other subjects of the elementary school. The impracticability of such a procedure should be evident. Therefore, the first requirement for a workable plan is to keep the pupils together as a class and to have the group progress uniformly with the textbook.

Assuming the class progresses at a uniform rate in the textbook, the problem of

providing for different abilities depends upon differentiation of the subject matter and of the method of dealing with the subject. The content of the course of study must be patterned to meet different levels of abilities. For the slow learner the criterion of social utility in arithmetic should be applied rigorously to such topics as common and decimal fractions. The slow learners would not be required to add unlike fractions in symbolic form unless one of the denominators is a common denominator. The fast learners would master a complete treatment of fractions. Division by a fractional divisor would not be part of the curriculum for the slow learners. It is much more important for the pupil to learn a few generalizations which apply to a process than to learn a mass of specifics which are factual and unrelated, and which will soon be forgotten. The teacher must accept the philosophy that all pupils in a class cannot understand all of the material in a textbook in arithmetic.

Ways of Differentiating the Level of Operation

The greatest possibility for providing for ability groups consists in differentiating the level of abstraction at which a pupil operates. There are five ways of differentiation of the level of operation. They are:

1. Use of materials
2. Different algorithms
3. Estimation
4. Solution of problems
5. Discovery of mathematical principles.

It has been mentioned how pupils can operate at different levels of abstraction by the use of materials. A pupil can find the sum of such fractions as $\frac{1}{2}$ and $\frac{1}{3}$ by use of fractional cutouts almost as easily as he can find the sum of $\frac{1}{3}$ and $\frac{1}{3}$. He may not understand the principle of finding a common denominator, but he knows how to find the sum of the fractions $\frac{1}{2}$ and $\frac{1}{3}$. The slow learner can use exploratory materials or visual aids to solve examples

which the fast learner would solve by using symbols. This differentiation of the level of abstraction of operation is an effective means to use to enable the teacher to deal with the same topic or process in daily work. When a pupil uses the kind of material which enables him to find the solutions of situations new to him, he feels secure and confident. Security and confidence are the two essential ingredients for growth in arithmetic. The use of different materials is effective from the standpoint of learning by the pupil and is easy for the teacher to administer.

The second means of differentiation of operation pertains to the learning of algorithms (ways of performing an operation). If teachers demand of all pupils the same procedure of performing a process, such as multiplication or division, provision for meeting differences in ability must be curtailed. It is not possible to have pupils operate at different levels of abstraction if each process must be performed in the same manner. We can illustrate how an algorithm, such as that for division by a two-place divisor, can be differentiated. The entire class would function as a whole in learning the meaning of division and how to divide by such divisors as 10 or multiples of 10. Examples of this order give a minimum of trouble for either the slow learner or the fast learner. It is in examples in which the estimated quotient must be corrected to find the true quotient that the slow learners encounter greatest difficulty in finding the quotient. Here the work should be differentiated to meet the abilities of different groups.

Suppose we consider dividing by a divisor in the teens, such as 16. The example, $16 \overline{)912}$, is difficult for slow learners if the conventional algorithm is used for finding the quotient. The slow learners should make a table by repeated doubling of the divisor so as to give 2, 4, and 8 times that number. For the divisor 16, the table would include 2×16 , 4×16 , 8×16 , and 10×16 . Since 5 is half of 10, the student also should include 5×16 in the table.

The table for the divisor 16 would be as follows:

$$\begin{aligned} 2 \times 16 &= 32 \\ 4 \times 16 &= 64 \quad (2 \times 32) \\ 5 \times 16 &= 80 \quad (1/2 \times 10 \times 16) \\ 8 \times 16 &= 128 \quad (2 \times 64) \\ 10 \times 16 &= 160 \end{aligned}$$

From the table, the pupil can use the values given to estimate each quotient figure. In the example, the pupil can see that the first quotient figure would be 5 tens. The second quotient figure could not be 8, hence he would try 7 which is the correct figure.

$$\begin{array}{r} 57 \\ 16 \overline{)912} \\ \underline{80} \\ 112 \\ \underline{112} \\ 0 \end{array}$$

The pupil does not need to multiply by a number greater than 2 in making the table. Then, too, he should discover something about the systematic way in which a table can be made. By this method he does not need a great amount of drill on the difficult multiplication facts which always are a marked source of error in division with a two-place divisor. He learns a pattern which will apply to all examples in division.

The group consisting of those of average ability would divide in the conventional manner. The pupil would find the correct quotient figure by an accepted usage of the guide figure. The fast learners would use a variety of ways of finding the quotient. A pupil in this group should be encouraged to make discoveries about the numbers to be divided. In this example both dividend and divisor are divisible by 4. Therefore, it would be possible to divide both 16 and 912 by 4. The resulting fraction, or example, would be $4 \overline{)228}$. This division could be completed to give a quotient of 57. The fast learners in arithmetic do not need a great amount of practice in the mechanics of the algorithms. These pupils should be encouraged to discover new and varied ways of finding a solution. Of course this group would always estimate the quotient to show that the answer is sen-

sible. Since the divisor, 16, is between 10 and 20, the quotient must be less than 90 and more than 40. The answer 57 is sensible.

In each group the emphasis should be placed on the understanding of the process and not on the number of examples solved. The plan is easy to administer and all of the material essential for its execution can be taken from the textbook.

The third plan of differentiation of operation consists in estimation of an answer to see if the answer is sensible. When a pupil estimates an answer he must be able to think intelligently with number. If the criterion of social utility is applied to determine the subject matter of the slow learners, even these pupils can approximate an answer to see if it is sensible. On the whole it is predominantly the fast learners who achieve well in estimation of answers.

Frequently, the methods for rounding off numbers for estimation follow no fixed pattern. This past summer the writer noted the sign at the entrance to the famous Rose Garden at Los Angeles. The sign stated that the garden contains 144 varieties of roses and 15,000 plants. A minimum of intellectual curiosity would urge one to find the average number of plants for each variety. This problem was given to a group of teachers studying in a nearby university. Upon questioning, it was discovered that all of the students rounded off 144 to 140 and then divided. This pattern of thinking follows the conventional way of rounding off numbers. A more intelligent way to solve the problem would be to think of 144 as 150 because of the number to be divided. Then the pattern of thought would be as follows: " $10 \times 150 = 1500$; $100 \times 150 = 15,000$. The average number per variety would be slightly more than 100 roses."

The fast learners should be able to estimate if an answer is sensible in such work as division of fractions. The pupil should be able to state if the quotient should be more than 1 or less than 1. In

the example, $\frac{3}{4} \div \frac{2}{3}$, the quotient is $\frac{9}{8}$. If the wrong term is inverted in performing the algorism, the answer will be $\frac{8}{9}$. The fast learner should discover that the answer $\frac{8}{9}$ is not sensible because the quotient must be greater than 1. There are two basic principles which apply to the quotient in division. They are:

1. A larger number divided by a smaller number gives a quotient of more than 1.
2. A smaller number divided by a larger number gives a quotient of less than 1.

By applying these basic principles to division of both common and decimal fractions, the pupil is able to determine if an answer is sensible. The superior pupil can succeed in applying these principles. He is able to compare two unlike fractions, such as $\frac{3}{4}$ and $\frac{2}{3}$, and tell which is larger. Then he can predict that the quotient of $\frac{3}{4}$ divided by $\frac{2}{3}$ should be greater than 1 because $\frac{3}{4}$ is larger than $\frac{2}{3}$. It has been the writer's experience that a great majority of pupils in the elementary grades are not able to make sensible estimations about the quotient found by dividing two unlike proper fractions. Work of this kind is intended to enrich the program of the fast learners in arithmetic.

The fourth plan of differentiation of operation consists in the treatment of verbal problems. A time honored way of providing for the fast learners is to assign more problems to these pupils than to the slower learners. This plan is ineffective, especially if the verbal problems are similar in structure. If the problems deal with some unit, then this plan has merit because the problems are different. It is important for the teacher to understand that the number of exercises or of problems is not the essential element in providing for differentiation of levels of operation.

The fast learners should be required to give more than one solution to most problems. It is more challenging for a pupil to give three solutions to each of four problems than to solve 12 similar problems by giving one solution to each. A fast learner

encounters a challenge when he is required to give several solutions to a problem. He may not be challenged to give one solution to a problem, but he must do mathematical thinking to give several more solutions to that problem. We can illustrate the method to use by giving four solutions to the following problem: A car travels 9 miles in 10 minutes. At that rate, how far will the car travel in an hour?

One solution of the problem would be to multiply 9 by the number of 10-minute periods in one hour. Another method would be to find the distance traveled in one minute and then multiply by 60. A third method would be to make a table to include 9 miles in 10 minutes, 18 miles in 20 minutes, 27 miles in 30 minutes, and 54 miles in 60 minutes. Finally, a pupil might compare the given rate with a rate of a mile per minute. He would see that in one hour the distance traveled would be 6 miles less than 60 miles. The pupil who solves problems in the manner described develops insight into number. He discovers relationships among numbers. Instead of giving a double diet of problems of the same type, the fast learner needs to solve very few more problems than the slow learner, but the superior pupil would solve them at a higher level of quantitative thinking than the slow learner would be able to do.

The final plan of differentiation of operation consists in having the fast learner identify and state the basic mathematical principles governing a solution. Elsewhere the writer⁴ showed the basic mathematical principles which govern the operations in arithmetic. The fast learners should understand these principles and should be able to identify one or more of these principles in most daily work. The procedure can be illustrated in dealing with formulas for finding the area of a triangle and a trapezoid.

The formula for the area of a triangle is

$A = \frac{1}{2}bh$. Finding the area of a triangle by substituting numbers in the formula provides practice in multiplication. There is no challenge in work of this kind for the fast learner. This pupil should be able to identify the mathematical principles involved. These principles include:

1. The order in which factors are multiplied does not affect the product.
2. Multiplying by $\frac{1}{2}$ gives the same answer as dividing by 2, or multiplying by a number is the same as dividing by its reciprocal.
3. To multiply or divide two or more factors by a number, multiply or divide only one of the factors by that number.

The formula for the area of a trapezoid is $A = \frac{1}{2}h(a+b)$. The three mathematical principles which apply for finding the area of a triangle also apply here. Another principle which applies to the formula for the area of a trapezoid is as follows: If an indicated sum is multiplied or divided by a number, each term within the indicated sum must be multiplied or divided by that number. The pupil who is able to identify the mathematical principles which govern the operations of arithmetic understands the subject. Arithmetic is structuralized. Most pupils do not understand the structure. It is certain that only those who succeed well will understand the complete structure of arithmetic. The subject must be so taught that these pupils have the opportunities for discovering the structure of the subject.

The first objective of enrichment seems to have been forgotten. This principle states that the pupil should learn to work independently and learn directly from books and periodicals. The plan discussed gives a minimum of consideration to the value of independent study.

The school library should be the heart of the enrichment program for the fast learners. This library should contain many interesting books dealing with some phase of mathematics. The pupils should be permitted to read the books at home and also during periods of inactivity during the

⁴ Grossnickle, Foster E. "Teaching Arithmetic in the Junior High School," *Mathematics Teacher*, 47: 520-27; December, 1954.

school day. Such books as *How Much and How Many?*, *Flatland*, *The Wonderful Wonders of One, Two, Three*, *Number Stories of Long Ago*, *Fun with Figures*, and *Winter Nights Entertainment* represent the kinds of books which should be available for independent study for those who excel in arithmetic.

Summary

Current methods of enriching the program in arithmetic consist in acceleration, segregation, and enrichment. The first and second means are predominantly administrative. The third means is good, but most of the activities in this category that now are employed operate without the framework of the textbook. This paper proposed that a greater amount of enrichment should come from within the framework of the textbook than is now achieved. The potentialities of the textbook in arithmetic are not adequately utilized. The program in arithmetic should be differentiated both in the curriculum and in the level of operation. The textbook usually will provide sufficient challenges for the fast learner if he uses a minimum of objective and visual aids, devises his own algorithms, estimates to see that answers to problems and examples are sensible, gives a variety of solutions to verbal problems, and identifies the mathematical principles which govern the basic operations. If the teacher follows in part or the whole of the proposed plan for enriching the work of those who excel in arithmetic, there is good reason to believe that these pupils will attain still higher levels of achievement in this subject.

EDITOR'S NOTE: Mr. Grossnickle proposes that we give more attention to the methods of learning employed by those who excel in learning arithmetic. He rightly condemns the practice of teachers who assign more of the same work to pupils who already excel in computations and problems. He suggests requiring the more able pupils to solve problems by several methods. Such work should lay the basis for broader understanding and for "transfer." It may be a little disconcerting for us to realize

that in any normal grade we are apt to find a few pupils who may excel even their teacher in intelligence. It is a challenge to know how to guide their learning for maximum present and future dividends.

Mr. Grossnickle agrees with other leaders that the aims of arithmetic are best served by flexible grouping growing out of whole class participation in the early stages of the development of a topic. The more able pupils soon move to level of abstraction while the slower continue slowly but surely with objective materials and simpler numbers. Perhaps it is better for the less able to understand fewer and simpler concepts than to attempt to learn computations by the current and conventional adult algorithms.

PUZZLES

Three Boys, Don, Jack, and Bill, ate their lunch together and spent 37¢, 49¢, and 49¢ respectively for which Bill paid with a \$5 bill with the agreement that they would adjust the proper amounts later. In change Bill received three one-dollar bills, a half dollar, a dime, and a nickel. Don has only a half dollar and 2 pennies and Jack has a single dollar bill. How can they make the proper change to each pays his correct amount?

In mythical times a man proposed to build a sunshade around the earth at the equator. He wanted it erected six feet from the earth's surface so that people could walk under it. How much material should he order so his sunshade band would go all the way around the earth? He asked the wise men of the town and obtained many different answers. Would a strip 40 feet longer than the circumference of the earth suffice?

How can you use the figure 3 four times and show the value 10?

$$3 \times 3 + \frac{3}{3} = 10$$

Puzzles like this are easy to construct. They are intriguing to many pupils. They can be made easy or difficult and they may have real value in extending understanding. How can you combine the digits 1, 2, 3, 4, 5 for a value of 5?

Implications of a Guidance and Counseling Program

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THE MAJOR PURPOSE of this discussion is to cite background thinking, basic assumptions, and techniques that have grown out of a program of Guidance and Counseling Testing introduced a number of years ago. The Detroit Public Schools foster and supervise this continuing program of periodic testing through the Division of Instruction and the Department of Instructional Research. Although the program cuts across a number of subject matter lines (except in the introductory background statement), I will attempt to point my remarks to our major interest—the testing program and its implication to mathematics instruction and programming.

Traditional Testing and Questionable Assumptions

In the past, testing has most frequently been associated with unhappy emotional experiences on the part of both teachers and students. This has been due to the fact that testing programs, as such, emerged from the administrative office and were used for making comparisons between students, classes, and schools. By and large, this practice resulted in a rather meaningless program in which tests were administered in a prescribed manner, scored, tabulated, and then returned to the central office. In terms of the individual, mathematics teachers had very little opportunity to use the students' test results under this practice.

Planning and administering such a testing program through the central office invariably resulted in inter-school comparisons and often between classes in the same building. This led to the rather

vicious idea that the teacher was being measured rather than the student. In this setting it is little wonder that teachers shied away from any comprehensive testing program.

Time will not permit elaboration on the unfortunate assumptions of testing programs in the past. However, it is necessary to mention a few to make the transition from the traditional testing program to the present one to be considered. It is felt that testing programs based on faulty assumptions have definitely inhibited teachers from adequately providing for individual differences to meet the needs of their students. A few of these assumptions which may have handicapped teachers are listed. However, they are not listed in any order of importance.

1. The first assumption was that the best teachers were grade or subject specialists with methods and techniques solely adaptable for a particular area or level. This assumption has been rather hard to eliminate though research studies have repeatedly revealed the range of achievement at any grade level to be four years or more.
2. The second assumption was that a course of study implied uniform administration to all students.
3. The third assumption was that a testing program determines whether the student should be allowed to continue to the next rung of the educational ladder. This assumption has primarily emphasized achievement testing for promotional purposes.

4. The fourth assumption was that if instruction is varied and intensified, all students can be brought up to "standard."

Perhaps the abuse most frequently made of test scores was in their isolated use to the exclusion of all other pertinent information. Many times students were counseled on the basis of a single intelligence quotient without reference to any other information. In the same manner, students were promoted or failed by a single raw score in arithmetic or algebra. Since schools are graded and perhaps will remain so for some time to come, the basic assumptions cited have stifled teachers in their attempt to adjust instruction to individual differences.

In summary, it should be emphasized that tradition, promotion policy, and the intent and purpose of testing programs have often stood in the way of, or at least handicapped, teachers in providing for individual differences. The idea of individual differences is not new; in fact, it is somewhat embarrassing to mention it for research established this principle years ago. Although the fact or principle is well established, the cause of the great variations is still being argued by two schools of thought—the hereditarianists and the environmentalists. In general, teachers have not taken sides in this controversy as the best instruction seems to result from teachers who accept both factors to be important. In the past, testing programs have been geared to the philosophy of homogeneous rather than heterogeneous ability grouping. Although both methods rely heavily on a testing program, the intent and purpose is vastly different. The major purpose was the attempt to make the classes homogeneous with respect to general ability and then standardize these classes by uniform assignments and textbooks. Confusion is apt to arise at this point, as in other fields of thought, from failure of different individuals to use terms in the same sense. For example, "student

grouping" to some individuals means "ability grouping," sometimes "interest grouping," and at other times it means "grade grouping." However, it is beyond the scope and the intent of this discussion to relieve the general confusion in regard to the term, "grouping." For limited reference it should suffice to say, "ability grouping" is used synonymously with homogeneous grouping; the classification of students within a particular grade will be regarded as "grade level"; and heterogeneous grouping will be in regard to intelligence ratings and not to all ages from the five-year-old to the eighteen-year-old who have different levels of achievement from kindergarten to the high school senior.

Guiding Principle and Basic Assumptions Underlining Our New Program

With the cited limitation inherent in our traditional testing program, and the fact that educators were not agreed as to whether ability grouping provided for individual differences in the classroom, it was decided to revise and redirect the purpose and objectives of our testing program. This was undertaken by a large committee of teachers, principals, supervisors, and division heads. The guiding thought seemed to be that measurement of mental abilities, achievement, personal characteristics, and interests offer the most solid basis on which pupils may be assisted in their future educational pursuits. Further, that test results gain greater potential value when they are used in a supplementary fashion with other pertinent information concerning the students. Students should no longer be counseled or grouped on the basis of a single intelligence quotient. Using test results with other data means taking into consideration *physical and health factors, extracurricular activities and interests, environmental factors, home and family situations, personality characteristics, and special abilities and disabilities*. A comprehensive

cumulative record should be started at the third grade level and should follow the students through the twelfth grade. *This does not refer to permanent office records, but to a record from that is available to every mathematics teacher who desires to use it.* The role of this cumulative record is that of a rudder to protect the raw test scores from drifting aimlessly about, without direction of purpose. *Any testing program that attempts to utilize test information without a cumulative record readily available and in the hands of the mathematics teacher will find its purpose sterile of any true value.*

With the above thinking as a base for direction and purpose, the committee formulated a Guidance and Counseling Testing Program designed to furnish elementary, junior high, and high schools with information that would enable teachers and counselors to be of greater assistance to students in Grades 5 through 12. Time will not permit me to elaborate on the principle and the basic assumptions growing out of the committee's work. However, it is necessary to mention the major premises on which the program grew. They are as follows.

The Principle of Individual Differences

Teachers may no longer ignore the fact that some students can never succeed in ordinary school work, or that students with low ability *may be able to work successfully in a different type of educational environment.* The low-ability student is too frequently discovered only through failure. Instead of being counseled and encouraged to master the work at his level of operation, he is often taught to fail. On the other hand, the high-ability student is too often neglected by the teacher's inability to recognize his potential. *Thus it is the responsibility of each mathematics teacher to discover and record the strengths and weaknesses of his students.*

Besides the principle of individual differences, the testing program is based on five rather basic assumptions.

The first assumption is that the major purpose of testing is to improve the quality of instruction by helping mathematics teachers make their offering more appropriate for students. This assumption does not imply that educational tests serve no other purpose.

The second assumption is that each student be given the opportunity to progress as rapidly as he can, or as slowly as he must. Education is a continuous process in which, by skillful guidance and counseling, the development of the student is guided on the basis of his potential and his previous growth. If maximum growth of each student is the aim of education, then tests or measures of the students' abilities, readiness levels, and progress must be recorded.

The third assumption is that the teachers will not restrict their testing to the subject matter areas exclusively. These subject matter areas should be augmented by measurements of the students' personal and social adjustment with a view to promoting maximum growth and development. The mental health of a pupil is the responsibility of all teachers—mathematics and otherwise.

The fourth assumption is that a testing program cannot be decreed by administration without the understanding or participation on the part of the teachers and counselors. If a testing program is to succeed as a functional part of the educative process, the purposes must be formulated and accepted by all—teachers, principals, supervisors, and administrative heads. This is thought of as in-service education of school staff.

The fifth assumption is that teachers must accept students where they find them. For the purpose of this discussion, this is defined as intellectually, academically, socially, and chronologically. Any instrument or test is justified if it affords a teacher additional information about a student. The more a teacher knows about a student, the better instructional guidance and counseling service he can render.

Although there were other assumptions made which perhaps played an equally important role in the formation of the present program, the five assumptions mentioned are sufficient to show the transition from the traditional testing program to the suggested one.

Guidance Counseling and Testing Program

The Guidance and Counseling Testing Program now includes all students in the fifth, seventh, and ninth or tenth grades. The testing periods are the first week in October and the second week in February. There is considerable advantage in administering the tests early in each semester. First, it provides the teachers with pertinent information concerning each student so that individual and class needs may be identified. Second, in giving the tests in the same month each year the gains and losses for each student may be compared to a year's educational growth without mathematical computation. This scheduling enables teachers to have the following information concerning each pupil:

1. An arithmetic and reading test score expressed in grade level.
2. A student's background form.
3. A letter rating of general intelligence.
4. An Algebra Aptitude Test score (eighth grade level only).

This information is the minimum required on the program. In order to make the information uniform throughout the city, this phase of the program is compulsory. However, the value of the testing program to any particular school depends entirely upon the effort made by that school to benefit from it. To merely participate in testing accomplishes nothing. Information obtained from the tests must form the basis for future action. What is actually done must be done by teachers and counselors in each school. However, *the Guidance and Counseling Records which*

are on the teacher's desk are a constant reminder to do something about individual differences.

Each of the tests and evaluative instruments used in the program are briefly described as follows:

Basic Arithmetic Skills.—This test is composed of three parts. Part I is concerned with vocabulary and fundamental knowledge. The student's performance on this section of the test will depend in part upon the scope of his technical mathematical vocabulary and in part on his ability to state what he is doing. No emphasis is placed upon mere rote learning of rules as such. Part II measures computational skill in the four fundamentals as applied to whole numbers, fractions, percentage, and decimals. Part III calls for the direct solution of problems. These problems present a fair sampling of the social situations in which mathematical skills are needed.

Silent Reading Comprehension.—This test consists of two parts. Part I is concerned with general reading comprehension. Part II is a test of word meaning, and measures the scope of the student's reading vocabulary.

Student's Backgrounds.—This inventory is designed to give teachers and counselors pertinent information which will help them to better understand student problems. They include vital information about hobbies, vocational and educational interests, reading and listening interests, home conditions, and recreational activities of the pupil. These inventories are easy to administer and provide the teacher with information for guidance.

Algebra Aptitude Test.—This test is designed to furnish information in determining a student's probable success in ninth grade algebra. The test consists of four parts:

- 1.—Arithmetic Reasoning
- 2.—Arithmetic Computation
- 3.—Formulas and Equations
- 4.—Number Series

Although the test is not a perfect measure of prediction, it is better than mere opinion or guessing.

At this particular time, the articulation program between eighth grade graduation and high school entrance is improving rapidly. Mathematics teachers are agreed that the transition from eighth grade mathematics to ninth grade algebra should not be made by all pupils.

Ever since general mathematics was introduced as an alternate course to algebra in the ninth grade, there has been the problem of how best to select the right students for these courses. It is believed that greater instructional efficiency is achieved if certain pupils are intelligently guided away from algebra and others guided or counseled into taking the course.

In brief, the following procedures are used in carrying out the algebra aptitude counseling plan. During the tenth week of the eighth grade, all 8A students are given the Algebra Aptitude Test by their respective mathematics teachers. When these tests are scored, the results are recorded on the pupil's Guidance and Counseling Record along with the results of his other special eighth grade tests. When this is completed, the records are forwarded to the high school mathematics department head and the counselor for their consideration in programming.

Most schools using this method have found it desirable to acquaint individual pupils with their own aptitude scores as soon as possible. These scores are then recorded on the pupil's 9B Curriculum Selection Sheet and taken home to his parents for approval. The counselor's suggested recommendation for algebra or general mathematics is written on the back of this form. It is strongly suggested that the pupil follow the recommendation. However, if a pupil has a low percentile score and it is his parents' wish that he should still take algebra in spite of the warning of possible failure, he is permitted to do so.

Since no plan is infallible and other fac-

tors contribute to failure in algebra besides ability, the student's progress is carefully observed during his first semester's work. When a teacher finds a student doing poor work, it is suggested that he transfer to general mathematics at the close of the semester without losing credit for the semester's work. Nearly all such pupils accept such an "out" to their mathematical difficulties.

Summary

The following are a few of the desirable outcomes due to the redirection of our testing program over the past three years.

1. The program has provided teachers with uniform test data and information concerning their students. The important consideration is that the cumulative record is in the mathematics teacher's possession and not in the permanent office file.
2. The program has provided regularity in testing schedule and comparable test data. Teacher's planning and counseling is more effective when based upon a regular evaluation of student growth and development. The student's self-appraisal is enhanced when based upon knowledge of his progress over approximately equal periods of time.
3. The program has provided teachers with the opportunity to analyze both strengths and weaknesses in individuals and classes at various grade levels. *Upper grade teachers have found it incorrect to assume that all incoming students have mastered the skills and fundamentals.* As a result, many refresher courses in mathematics have been introduced at the tenth and eleventh grade levels. The program has supplied the necessary data for mathematics teachers to better provide for individual differences and assume the role of counselor in future planning for their students.

4. The program has pointed out that the development of the fundamental skills is not the sole responsibility of any one teacher or at any particular grade level of the educational system. All grade levels must meet the developmental needs of students. The elementary school has the responsibility of developing the fundamental skills in mathematics to an optimum degree. The secondary school must also assume this responsibility of accepting students at various levels of achievement. They too must share the responsibility of promoting growth and development. Thus the teaching of skills and fundamentals becomes a continuous responsibility of all teachers and all grade levels in a school system.

Our research over a number of years has shown the program to be relieving algebra failure, and at the same time producing a better over-all understanding of growth and development of students.

EDITOR'S NOTE: In recent years guidance within a school system has become more important and this service in many schools has been organized in such a way that pupils are served as individual human beings and not just a case number. This is a different type of guidance than vocational guidance and yet it is integrally related thereto. Dr. McDauid has expressed a number of fine ideas which we as teachers at any level should consider. To do our best job, we need to know a good deal about pupils. It is the teacher who is the direct contact with the pupil and he is the one who should have the records for guidance. But a good teacher will know a great deal more than can be shown in even an informal record. And it is the teacher's human understanding of pupils that provides a wholesome atmosphere and attitude. Too often written records are accepted as absolutes when probably such an item as an I.Q. should be recorded as perhaps 105 ± 8 . Dr. McDauid also has pointed out how important it is not to assume that pupils in grade six know all the arithmetic of the preceding grades. It is the responsibility of teachers at all levels to develop greater depth of understanding and better facility of work with all their pupils.

The Hundred Board

JESSE OSBORN

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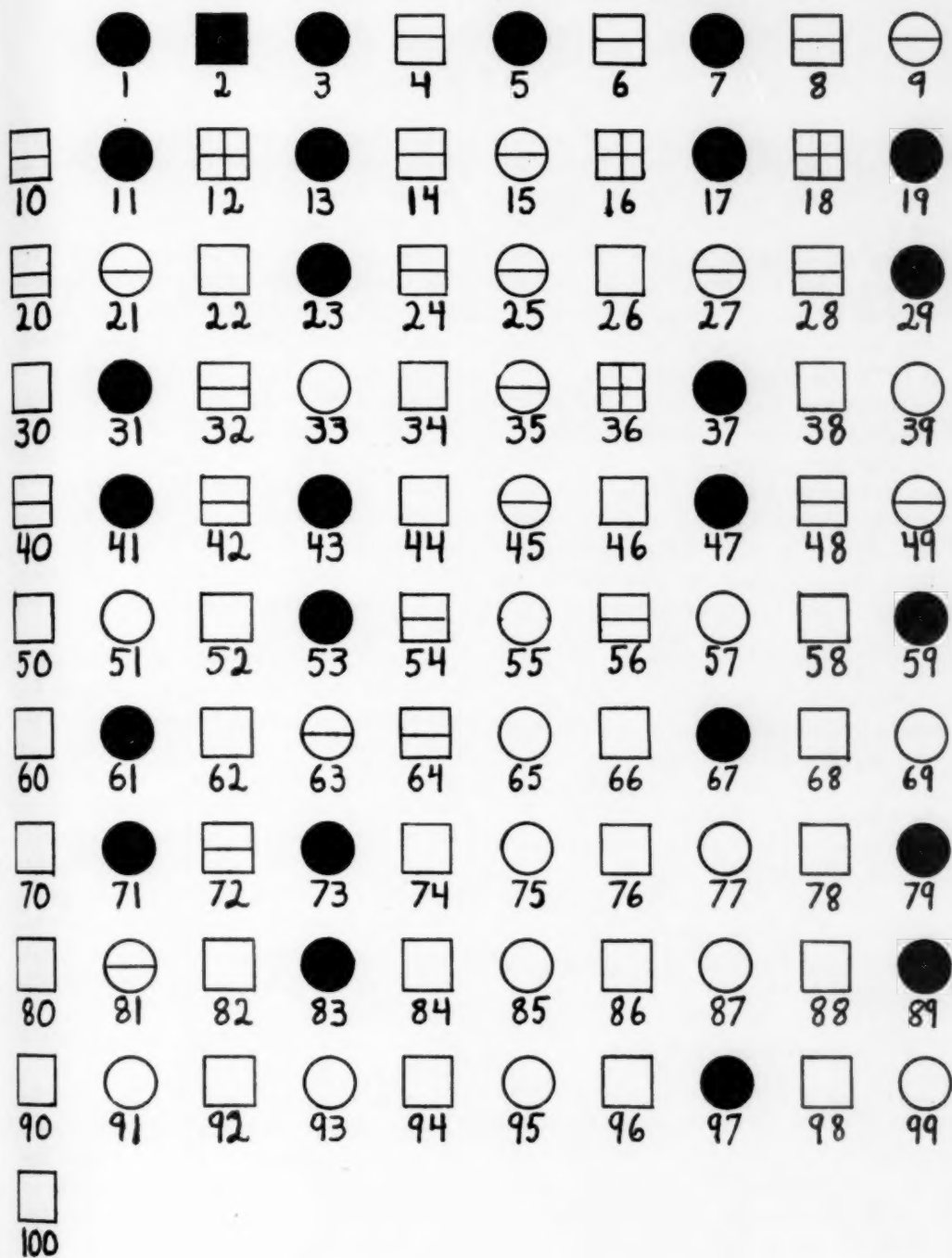
MASTERY OF THE WHOLE NUMBERS to 100 is basic to everything that follows in arithmetic, and in most other mathematics. The chart shown here is adapted from a board that has helped many pupils toward such mastery.

Circles represent odd numbers; squares represent even numbers. Black circles and squares represent prime numbers; white ones represent composite numbers.

Circles or squares that represent parts of basic multiplication (division) facts have lines across them. One line means the number has one distinct pair of factors, as $15 = 3 \times 5$. Two lines means the number has two pairs of distinct factors, as $12 = 2 \times 6$ or 3×4 .

This Hundred Board has been most useful when it is a real board rather than a chart. The back can be plywood from two to four feet square. The circles and squares are sawed out of other lumber. Their edges are rounded and they are sanded smooth. They are painted black or white to fit the plan of the board, and are glued in place. Saw cuts are used to mark the circles and squares that are parts of basic multiplication (division) facts.

With this Board it is easy to see the pattern of odd and even numbers. Prime and composite numbers stand out. Numbers that are parts of basic multiplication (division) facts are clearly marked. The pupil touches and counts by 2's, 3's, 4's, . . . By spanning or measuring off with two hands, he learns to put groups together and to take groups apart. Basic facts cease to be just symbols on paper. They can be proved with things he can see and touch.



The Intangibles of Arithmetic Learning*

JOHN R. CLARK

New Hope, Pa.

BEFORE UNDERTAKING A DISCUSSION of the intangibles of arithmetic learning, we should comment upon the tangibles. These include (1) ideas (meanings or concepts), (2) skills (computational techniques), and (3) reasoning or problem solving. We always recognize these tangibles in our thinking about the *objectives* of *arithmetic learning*.

Arithmetic learning is a complex of these tangibles and intangibles. But what are the intangibles? This discussion assumes that they are the "feeling" aspects of learning. We may call them attitudes, dispositions, interests, likes, dislikes, etc. The intangibles are the pre-cedents, the concomitants, and the resultants of learning experiences which deal directly with the tangibles.

We have given much thought to the teaching of the tangibles. We are relatively sure about the conditions for success in building concepts, acquiring skills, and the nature of problem analysis. We have been preoccupied with improving the learning of these.

We assume that these intangibles are important, very important indeed. But we write and talk less about guiding the acquisition of them.

Intangibles Are Real

After the above prefatory statement we come to the problem—making sure of the intangibles. The writer, with some confidence, offers the following observations:

1. These intangibles constitute the most permanent aspect of one's learning. The learner may forget his *facts*; his skills may atrophy with disuse; even his concepts

may erode with the passing of time; but the way he feels about the subject lives on and on with him.

2. These intangibles tend to spread out, to transfer to, other learning situations. They become potent factors in the moulding of the learner's personality and ego.

3. For any learner, the primary determiner of these intangibles in success (or failure) in his experience with the tangibles. By and large, he learns to *like* mathematics when he is aware of his success in dealing with mathematics; he learns to *dislike* the subject if he finds that he cannot manage it. Thus it is imperative that the learner experience success more often than failure! The majority of his goals must be *attainable*.

Now the wide range of ability and achievement in a class makes it difficult for the teacher to set up learning goals with the *optimum* degree of challenge and attainability for each pupil in her class. She knows that a given goal may be so difficult that Tom is unable to achieve it, and hence will become discouraged. She knows also that the same goal may be so easy that Ben finds no challenge or adventure in undertaking it, and hence becomes bored and disinterested. Truly she is confronted with a problem, providing for the range of ability and achievement of her pupils.

Before going further with the problem of assuring a favorable attitude toward arithmetic learning, let us restate our major assumption: There is a feeling accompaniment to learning. Desirable feeling accompaniments are dependent upon successful efforts. Undesirable feeling accompaniments certainly follow lack of success. We are not saying, however, that success means always getting of right answers in computation or reasoning.

* Summary of an address at the Washington meeting of the National Council of Teachers of Mathematics, December, 1955.

Slow or immature learners *must* experience success most of the time. Bright mature learners must recognize that the attainment of the goal involves some mental risk, some hazard. Their diet requires some hard-to-crack nuts, with occasional very hard ones. We repeat, the goals should be attainable and of optimum difficulty.

Returning to the teacher's problem of providing for the range of ability and achievement of her pupils, we shall consider some suggested solutions. Let the class be composed of eight-year-olds.

One likely procedure is that of sub-grouping within the class, according to ability and/or achievement. Obviously this sub-grouping greatly reduces the spread or range of differences, even though it does not produce homogenous groups. One of the sub-groups, Group A, (assume there are only two) will be less mature, less able to think with numbers, less able to generalize number relations, less able to make deductions from symbols, than Group B. Group B will be less dependent upon manipulative aids, more able to "think out" ingenious ways of solving a problem, and will require less practice in order to retain and recall.

The differences between Group A and Group B will increase, if their goals during the year are appropriate to their levels of maturity. Even though both groups deal with the same arithmetic subject matter (as they probably should), their approaches to their learning will differ. Group B will achieve greater depth of learning, will become more inventive and more resourceful. The teacher, however, is terribly concerned that the pupils in both groups "feel good" about their learning—be aware of their success, be able to achieve on their maturity level.

In many schools in which there are two or more classes of eight-year-olds, sectioning by classes is made according to ability and/or achievement, thus decreasing the heterogeneity in the resulting classes.

A second approach to the problem of

achieving maximum success for every learner presupposes neither sub-grouping within a class nor sectioning by classes. The writer calls it the Springfield (Mass.) plan, because so far as he knows, it was first described by the Springfield Arithmetic Committee.

Unfortunately, in teaching eight-year-olds, instruction too often is pitched to the middle half or two thirds of the class. The lower fourth (or sixth) of the class soon acquires a persistent defeatist attitude resulting from inability to achieve the goals set by the teacher. The upper fourth (or sixth) of the class soon loses interest in the subject and, because it lacks the necessary challenge and depth of learning, does not achieve its potential.

Cultivating the Intangibles

We now venture some specific suggestions for cultivating these very significant intangibles among our eight-year-olds.

Take adequate care of the tangibles, and the intangibles will largely take care of themselves.

1. Become well acquainted with the learner's maturity, ability, and interests. Judge his response in terms of his maturity.

2. Keep in mind that there are many correct, even though not equally mature, ways of thinking about number relationships and reasoning.

3. Consider that the class period is primarily a thinking experience, not a period in which learners are to respond in fixed, rigidly prescribed, patterns.

4. Be generous in commending both mature and immature learners for the correctness and validity of their thinking. Occasionally say "That is wonderful thinking," or "Is that good thinking?", or "Whose thinking do you prefer?"

5. Remember that you as a teacher can learn new ways of thinking from your pupils.

6. Encourage the learner to tell how he could "think out" the problem without the help of the diagram, or fraction cutouts, or the bead frame, etc.

7. Be particularly interested in the learner's growth in maturity as well as in the correctness or incorrectness of his answers. Be as much concerned with his thinking as with his answers. Expect and be prepared to take account of differences in maturity level and rates of learning.

8. Remember that learning by thinking is superior to learning by being told.

9. Consider memorizing for future recall the last stage, not the initial stage of learning.

10. Realize that learning is *thinking*. Successful thinking offers the best assurance of independence, self-reliance and confidence in meeting new situations. The old adage that "nothing succeeds like success" is profoundly true.

EDITOR'S NOTE: Mr. Clark calls to our attention the very important "feeling" aspects of learning arithmetic. It is these feelings and attitudes that not only give depth and dimension to learning but also provide the more permanent ties which a pupil forms with the subject and the teacher. In many schools it is easy to sense a pleasant "educational atmosphere," the kind in which Mr. Clark's intangibles are fostered. But a good atmosphere is more than pleasant, it is also stimulating and this stimulation differs for the different levels of pupils found in a single classroom. These intangibles often grow out of individual discovery of the type that leads to resourcefulness. It is still true that "honey will attract more flies than vinegar." Let us remember the intangibles as well as the tangibles in our teaching.

Time and Distance

In Biblical times the expression "3 day's journey" was common. In modern times however we have given distances for travel most commonly in miles. Now, with airplane travel becoming increasingly popular we are again speaking of travel distance in units of time such as "a 2-hour flight". Pupils in intermediate and upper grades might investigate various modes of expressing travel-distance. This can lead to the formula $d=rt$ and a study of the relationships of the three elements in the formula.

IS YOUR ATTITUDE SHOWING?

or

ARE YOU JUDY'S TEACHER?

The leader of an arithmetic workshop was giving a demonstration lesson in teaching children to do mental computation. Judy, a fifth-grader, gave a brilliant performance of thinking with numbers in a way hitherto untried by her. At the end of the lesson the leader asked Judy if, when she went back to her classroom, she would like to teach her teacher how to compute mentally.

Judy deliberated a moment, then thoughtfully responded, "That wouldn't be a good idea. She wouldn't want to learn."

On the Level

When a surveyor measures a distance such as the size of a city lot or a farm plot does he measure along the surface of the ground? What does he do if the terrain is sloping? If he measured along the slope what problems would arise in fitting plots together? Does a city lot on a hillside have more actual surface than a lot the same size on a level plot? Is there more grazing land on an acre of rolling land than on an acre of flatland? When grain production tests are made do the people who measure the plots correct for sloping ground so that test plots actually have exactly the same area of surface? It might be worthwhile to have pupils speculate on methods of measuring land when they are studying area. Ask them how they would measure the level distance on a plot that is sloping and then let someone find out from a surveyor how it is actually done. When pupils realize the difficulty of precise measurement they will understand why titles to land usually read like 85.6 acres, *more or less* and why plots laid out more than 100 years ago may have errors as large as 10 per cent.

Mathematical Background for Teachers of Arithmetic*

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Introduction to Present Conditions

ALL OF US have heard such expressions as "you cannot teach what you do not know"; or such vague expressions as "we teach children, not subject matter." These are typical of two opposing views regarding the professional preparation of teachers of arithmetic. The one view holds that methods and devices should be emphasized; the other puts a premium on the mastery of subject matter. The more modern view regards neither of these as wholly satisfactory . . . and places its hope for improved teaching of arithmetic on the development of a new generation of teachers who know both *what to teach* and *how to teach it*. The trend toward closer cooperation between mathematicians and educationists in a united attack on this problem is very encouraging.

It is my purpose to review briefly 1) some of the conditions which now exist regarding the preparation of elementary school teachers of arithmetic and 2) bring you the results of my recent study of the views held by selected leaders in the field regarding the major topics of mathematical content they consider basic to a required course for the preparation of elementary school teachers of arithmetic (Grades 1-6). Let us now take a look at:

* Presented at the Washington meeting of The National Council of Teachers of Mathematics, December, 1955.

State Minimum Requirements in Years of Training for Certification of Elementary School Teachers¹

In 1920, the model type of training program for elementary school teachers was the two-year program, the old Normal School Program. Since that time most State Normal Schools have expanded their various curricula, both in *depth* and *breadth*. They are now State Teachers Colleges—with a model program, a four-year elementary course.

At the time of Woellner and Wood's report (see the Table below) we find that 20 states required at least 4 years of professional preparation for the certification of their elementary school teachers. At the same time, 18 states required 2 years of preparation, 5 states only 1 year, and 4 states, 3 years—of professional training beyond the high school as minimum bases for certification.

No. of Years of Professional Study	No. of States
1	5
2	18
3	4
4	20
Not specified	1

¹ Adapted from Woellner, Robert C., and Wood; M. Aurilla, "Requirements for Certification of Teachers for Elementary Schools and Junior Colleges." University of Chicago Press, 1949 (10th Edition).

The teacher shortage during the recent war years (and at the present time) possibly has had some delaying influence upon demands for higher certification requirements. Of one thing we can be sure—as long as this acute shortage continues, instruction in all subjects—including arithmetic (or perhaps especially so in arithmetic) will not be up to par. The teachers who are presently returning to the classrooms after fifteen to twenty years of separation—are doing so for various reasons known best to the teacher species. Their added years have given them maturity and increased understanding of children—but frequently very little growth in knowledge of arithmetic. There is a crying need to offer these so-called “emergency” teachers our immediate help. Those of us who have been trying to do so—are impressed by their sincerity and eagerness for learning—and at the same time appalled by the pitifully meager background in mathematics upon which to build. This solution to the problem of how to deal with our “emergency” volunteers in the teaching ranks today becomes an important consideration in our reflections upon the necessary mathematical background for teachers of arithmetic. More about this later.

Arithmetic in the Various Teacher Training Programs of Our State Teachers Colleges²

In Professor Grossnickle's study of “The Training of Teachers of Arithmetic,” reported in the 50th Yearbook of the Society For The Study Of Education he found that 65% of the State Teachers Colleges' programs in arithmetic offer the same course in arithmetic for all teachers; 35% offer arithmetic in differentiated programs. Of the latter 35% offer arithmetic content designed for Grades I-III; 16% for Grades IV-VI; 13% for the Advanced Grades

VII-VIII; and 23% offer a general elementary course designed for Grades VII-VIII or IX. Grossnickle points out that, “We may undertake to prepare teachers for the elementary school in the ‘consolidated’ curriculum or we may undertake to prepare them to teach arithmetic and other social arts in a balanced program.” If we choose the specialization route of teacher preparation a student need not limit his preparation by studying only those problems of a certain grade level—say primary, or intermediate, or upper levels—but we should provide them with an over-all view and perspective of the work of the total program—and with special competence to deal most effectively with the problems likely to occur within the area of his choice.

The following table shows the types of arithmetic programs offered at various State Teachers Colleges as reported by Grossnickle.³

Type of Program	Number of Colleges	Percentage
Same for all teachers	84	65
Differentiated		
Kindergarten-primary	45	35
Grades (Kindergarten & Grades I-III)	45	35
Intermediate Grades (IV-VI)	20	16
Advanced Grades (VII-VIII)	17	13
General Elementary (Kindergarten & Grades I-VII or VIII)	30	23

High School Mathematics Requirements for Admission to State Teachers Colleges' Elementary School Curricula

It is obvious that the individual needs of arithmetic teachers in training depend upon the background in mathematics which they bring with them to the Teachers Colleges. For instance, if an entering

² Adapted from Table 3, p. 207, “The Training of Teachers of Arithmetic,” Foster E. Grossnickle. “The Teaching of Arithmetic,” 50th Yearbook, Part II, 1951 (NSSE).

³ *Ibid.*

student has had no mathematics beyond that of eighth-grade arithmetic—and in many instances is not competent even on this level—he will need basic subject matter course(s) in mathematics to provide him the competence, and confidence to guide the learning experiences of young children in their growth and development in arithmetic. What the nature of the college course or courses in mathematics for elementary school teachers should be will depend partially, at least, upon what preparation they have had in high school. To find out specifically the content of the high school courses taken by prospective teachers of elementary school arithmetic has not been subjected to critical study. Grossnickle, however, did report a sample study of the number of years of high school mathematics required for admission to various State Teachers Colleges.⁴ The following table reveals a tragic condition:

NUMBER OF YEARS OF HIGH SCHOOL MATHEMATICS REQUIRED FOR ADMISSION TO STATE TEACHERS COLLEGES' CURRICULUMS PREPARING TEACHERS OF ARITHMETIC

Kind of Mathematics	No. of Colleges	%
No Mathematics	98	76.0
Some form of Mathematics	31	24.0
One year of Algebra	2	1.6
One year of Geometry	2	1.6
One year of both Algebra & Geometry	5	3.9
One year of General Math.	2	1.6
One year of any kind of Math.	18	13.9
Two years of any kind of Math.	2	1.6

If 76% of the students enrolled in our various curricula for the preparation of teachers of arithmetic present *no* background in high school mathematics—and most often a very meager understanding of the fundamentals of arithmetic, there is little wonder that a one-semester teachers

course in arithmetic is so inadequate. Professor V. J. Glennon's study⁵ revealed that Teachers College students know only about 44% of the basic mathematical understandings on which he tested them. His observations at that time may be summed up in this quotation from his report: "it is hardly possible for teachers to help children grow in the understandings which they themselves do not possess." Those of us who attempt to teach courses in "The Teaching of Arithmetic" to persons with years of experience know how depressing it is to find such woeful lack of understanding of all but the very basic manipulative skills. The study made in 1952 by Jacob S. Orleans⁶ with student teachers and prospective teachers reveals conclusively the serious difficulties they have in explaining arithmetic concepts and processes. He points out the routinized procedures in practice and that probably, because of the lack of understanding possessed by these teachers, there is doubtless very little meaning developed through their teaching. He has found by personal conference with students and through class discussions that they are well aware of their lack of understanding of basic concepts, processes and relationships and their lack of ability to use arithmetic with confidence and meaning.

This situation is deplorable, to say the least, but is not one which will be corrected by itself. The teacher preparation in arithmetic must be dealt with promptly and wisely if we are to avoid having a new crop of teachers blindly leading the blind. Our hope lies in the creation of a realistic program of "Arithmetic for Teachers."

⁴ Glennon, V. J. A., "A Study of the Growth and Mastery of Certain Basic Mathematical Understandings on Seven Educational Levels," Doctor's Dissertation, Cambridge, Harvard University Graduate School of Education, 1948.

⁵ Orleans, Jacob S. "The Understanding of Arithmetic Processes and Concepts Possessed by Teachers of Arithmetic," Office of Research and Evaluation, Division of Teacher Education, Publication #12, N. Y., College of the City of New York, N. Y., 59 pp., 1952.

⁶ Adapted from Table 3, p. 207, "The Training of Teachers of Arithmetic," Foster E. Grossnickle. "The Teaching of Arithmetic," 50th Yearbook, Part II, 1951 (NSSE).

This means, of course, a demand for some preliminary high school mathematics before students enter the teacher training program, a realistically designed course or courses in mathematics for their college preparation, and improved certification practices. What the content of these courses shall be must be determined by 1) an analysis of what our best teachers now do; and 2) some frontier research into the kinds of mathematics our pupils need to meet the citizenship demands of this present era; and 3) continuous re-evaluation and study in the area of improvements in learning.

Some Previous Proposals for Improving the Preparation of Teachers of Arithmetic

In 1947 two National Agencies presented reports on their studies on the training of arithmetic teachers. The studies—and the reports which followed—were each made separately and independently. The "Manpower Report," made by the President's Scientific Research Board said this about the preparation of teachers of arithmetic:

"A professionalized subject-matter course emphasizing the use of mathematics in projects undertaken by children to learn the meaning of concepts is a minimum requirement for the training of mathematics teachers for the first six grades. Such a course should demonstrate the proper use of laboratory materials and appraisal techniques and should give the prospective teacher a clear notion of specific objectives for each grade level."

For the teacher of Grades VII and VIII this report makes the following observations:

"A vast amount of new material has come into the work of the pupils in these grades during the last thirty years. This material is not ordinarily included in the traditional sequential courses of the high school or of the college. Consequently, too many beginning teachers of seventh grade mathematics are expected to teach with-

out adequate time for planning that which they themselves have never studied in systematic fashion. The teacher of the seventh and eighth grades should have a minor consisting of mathematics courses especially designed to meet his needs. Few colleges now offer enough mathematics of this type for elementary teachers. The content of such courses is a matter that should be determined by a comprehensive study. In the meantime every teacher training institution should try to make available a course in the teaching of mathematics for these grades."⁷

The Commission on Postwar Plans of the National Council of Teachers of Mathematics made, in essence, these suggestions in this report:

Teachers preparing to teach arithmetic in the elementary school should study:

- 1) A good course in the teaching of arithmetic
- 2) One or more courses in the subject-matter background for the teaching of arithmetic in Grades I through VI
- 3) At least one additional mathematics content course for the teaching of General Mathematics in Grades VII and VIII.

The report also suggests that Junior High School teachers of mathematics should have at least a minor in college mathematics—including a year of study in general mathematics on the college level, one course in statistics, another in the mathematics of investment, and a professionalized subject matter to be taught in the Junior High School.

The Mathematics Background for Teachers of Arithmetic (Grades I-VI) (What are the important topics to be included?)

If the recommendations of these Commissions are sound—and if they are to

⁷ Manpower for Research, Vol. IV, Science and Public Policy. The President's Scientific Research Board. Washington: Superintendent of Documents, Government Printing Office, 1947.

culminate in appropriate courses, it is essential that we decide upon:

- a) What topics should be included, and
- b) The extent to which each topic shall be explored during the cadet's professional preparation.

To avoid presenting just one man's opinion regarding the mathematical con-

tent to be included in a background course for teachers of arithmetic (Grades I-VI), the problem was referred to a representative group of specialists in arithmetic.

The question form used and the poll results of the opinions of the specialists in arithmetic are printed below. The opinions have been expressed as per cents in each of the categories preceding the statements.

PART I

Directions: Indicate the importance you attach to each of the following topics in a course in mathematics to be required for undergraduate teachers preparing for positions in elementary schools—Grades 1-6. To the left of each topic write:

- (1) if you consider it *most important*
- (2) if you consider it *important*
- (3) if you consider it *optional*
- (4) if you would *reject* it.

Topics under Consideration

POLL RESULTS

(1)	(2)	(3)	(4)
20%	70%	10%	
70%	10%	10%	10%
90%	10%		
10%	40%	50%	
10%	60%	30%	
30%	70%		
90%	10%		
70%	10%	10%	10%
30%	50%	20%	
80%	20%		
80%	20%		
30%	40%	30%	
	50%	40%	10%
50%	40%	10%	
50%	30%		20%
40%	30%	30%	
40%	10%	50%	
10%	70%	10%	10%
	60%	40%	
	20%	40%	40%
	10%	50%	40%
	10%	30%	60%
		10%	90%
	50%	40%	10%

1. Improvement in the college student's arithmetic computation
2. Evaluation of the college student's arithmetic learning
3. Increased understanding of the number system (Hindu-Arabic)
4. High points in the development (historical) of our number system
5. Computation with approximate numbers (precision, possible error, etc.)
6. Measurement _____ (A) DIRECT _____ (B) INDIRECT
7. Concepts of operation
8. Discovery of rules
9. Checking results
10. Acquisition of power in reasoning (problem solving)
11. The inverse relationships within the operations
12. Number bases other than 10
13. Labeling of answers (to written solutions)
14. Postulates of operation (associate, commutative, distributive)
15. Alternate algorithms for the operations
16. Mental computation
17. Socio-Economic applications (home, market, finance, industry, etc.)
18. Algebra (formulas, equations, graphs)
19. Informal Geometry
20. Demonstrative Geometry
21. Trigonometry (numerical)
22. Analytic geometry (conic sections)
23. The Calculus
24. Elements of statistics

What topics other than those listed would you recommend for serious consideration by those teaching the course referred to above? Designate the relative importance you attach to each by writing 1, 2, or 3 before it.

Other Topics Suggested by the Jurors

1. Foundations of Number (Logical development)
2. Elements of Set Theory

3. Nomenclature

4. (Some other topics suggested for inclusion were implied in the list submitted.)

PART II

Directions: Place a *check mark* before the response of your choice.*

1. How much high school mathematics should be required of students preparing to take the course consisting of the topics listed in this questionnaire?

10% one year; 80% two years; 10% more than two years; _____ none.

2. How many semester hours of college class work would be required to treat adequately the topics you have checked?

a) suggested minimum: _____ (semester hours)

b) recommended : _____ (semester hours)

2. To the question, "How many semester hours of college class work would be required to treat the topics you have checked?" The responses are as follows:

A. Suggested Minimum:	a) A 2	sem. hr. course:	20%
	b) A 3	sem. hr. course:	30%
	c) A 4	sem. hr. course:	10%
	d) A 5	sem. hr. course:	10%
	e) A 6	sem. hr. course:	30%

B. Recommended:	a) A 3	sem. hr. course:	30%
	b) A 4	sem. hr. course:	10%
	c) A 6	sem. hr. course:	40%
	d) A 6-8	sem. hr. course:	10%
	e) An 8	sem. hr. course:	10%

3. Should the topics you have checked be treated from the standpoint of:

50% a) mathematical content?

_____ b) mathematical methodology?

50% c) both mathematical content and methods combined?

4. During which of the four years of college preparation should this course be taught?

30% 1st year; 40% 2nd year; 20% 3rd year; _____ 4th year.

10% Any time before beginning practice teaching in Arithmetic.

5. Would this course provide adequate background for teachers preparing for positions in grades 7 and 8?

10% yes; 90% no.

6. Should this course be under the direction of:

40% a) the Department of Mathematics?

10% b) the Department of Education?

50% c) both departments jointly?

* The per cents of responses falling in each category have been inserted in the blanks originally provided on the question form.

EDITOR'S NOTE: Mr. Snader's data tends to show that his "jury of experts" would like elementary school teachers to have studied mathematics for a minimum of six semester hours. But this is not the typical college mathematics. It is a mathematics that involves mainly the understanding of the backgrounds which an intelligent teacher of arithmetic needs in her job. This is not a methods

(Continued on bottom of next page)

The Abacus and Multiplication

M. VERE DeVault

The University of Texas, Austin

IN THE NOVEMBER, 1955, issue of this JOURNAL, Mr. Flewelling presented an excellent article on *The Abacus as an Arithmetic Teaching Device*. As pointed out in the Editor's Note following the article, "the greatest value in using an abacus comes from gaining insight into the number system and our modes of calculating."

The use of the abacus can make substantial contributions to the building of mathematical concepts at all grade levels. In the lower grades it may serve as a bead counter or as a fact finder and in the middle grades it may contribute to the child's understanding of the number processes.

Teachers who emphasize the use of many concrete teaching materials in arithmetic classes usually include the abacus as one of these aids. While much has been written about getting the abacus and other teaching materials into elementary classrooms, less has been written about how to use these materials in actual teaching situations. Thus, a few teachers know how to use the abacus in addition and subtraction but almost none of them use it in multiplication and division.

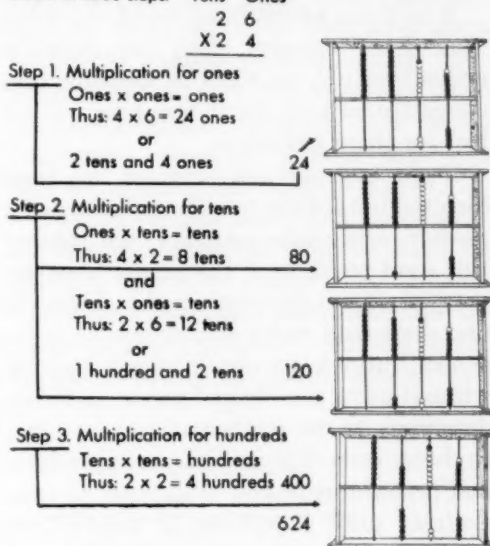
An understanding of the multiplication process will be facilitated through the use of the abacus as suggested by Mr. Flewelling. A next step in one's understanding can be made through a combination use of

the abacus and the lightning method of multiplication as discussed by Spitzer.¹

This method of multiplication is not suggested as a replacement for that illustrated by Mr. Flewelling. It is suggested rather as another use of the abacus for the extension of the teacher's and pupil's understanding of the multiplication process.

This use of the abacus is shown below.

The use of the lightning method in the multiplication of 2 two-digit numbers with the abacus is demonstrated below in three steps: Tens Ones



¹ Spitzer, Herbert F., *The Teaching of Arithmetic*, Boston, Houghton Mifflin Company, 1954, pp. 383-384.

course, it is a content course but it probably would have many aspects of method both inherent and concomitant. All too few colleges now educating elementary school teachers require this much mathematics. In fact a number of such schools require no mathematics and very little methods in arithmetic. Isn't it still true that "He who teaches all he knows must know more than he teaches." The newer arithmetic with emphasis upon thinking and understanding requires better educated teachers than does the old "drill-learn-forget" arithmetic in which the teacher was little more than a taskmaster. Mr. Snader quotes Mr. Grossnickle's figures showing that 76% of teachers colleges require no mathematics in high school prior to entrance to their training to become elementary school teachers. Actually the situation is not as bad as the figures seem to indicate. The Editor works in a college which requires no mathematics for entrance but only 5% enter with no high school mathematics, 12% come with one year, 33% come with 2 years, and 50% come with more than 2 years of high school mathematics. The factor of "will to learn" is often more important than the actual background of exposure to courses.

Testing the Attainment of the Broader Objectives of Arithmetic

HAVERLY O. MOYER

State University Teachers College, Plattsburg, N. Y.

TEACHERS HAVE ALWAYS GIVEN TESTS for one reason or another. Many times these tests have been homemade; sometimes they have been standardized. Both kinds have a place in the evaluation of any arithmetic program. However, if these tests measure only the ability to compute and to solve abstract problems then they can be criticized more for what they have not measured than for what they have, to some degree, measured.

If they can be criticized mainly for what they don't measure, then the question arises what do we want to measure that conventional tests don't and how will we test for these outcomes.

If the instructional program has been directed toward the broader outcomes of a modern arithmetic program, the teacher will want to know if the pupils have developed or are developing: the ability to find short-cuts, to see similarities, to make generalizations, to obtain some answers without using pencils and paper, to follow directions, to use mathematical reasoning, to choose correct procedures, to apply facts and procedures to new situations, to concentrate until a problem is finished, to extend what they have learned to the next logical step, to grasp quickly a problem and suggest a solution, and to evaluate objectively their own progress. If progress in these abilities is to be appraised the teacher will need to use more than the conventional teacher-made and standardized tests. He will need to use observational techniques and more unique ways of appraisal than have generally been employed to-date.

To illustrate this idea an example will be given. However, it should be remembered this is an example, only, and not a definite, complete test. This example

should suggest to teachers a way of adapting it to their specific purposes and the possibilities of developing other similar tests that can indicate a growth in the broader purposes of arithmetic teaching.

Suppose a teacher wishes to find out how well his class can use simple fractions when presented with an entirely new and somewhat unique problem. He may want to know: which pupils grasp a new situation fastest; which ones can follow directions of a particular type; which ones have the concept that a fraction which is a quantity, may be a unit as well as a part of a unit, and that a unit may be a fraction; which ones recognize the similarity of a second problem that is almost the same as a previous one.

To find the answer to these questions and others which the teacher could devise to suit his purpose and the maturation of the children, he could choose to use a protean puzzle. Children are usually intrigued with a puzzle and motivated to do their best to solve it if it seems at all probable to them that they may succeed. The teacher may need to make a check list first on which to record certain observations as the testing proceeds. Such a list can be made by placing the names of the pupils along one side of a ruled sheet and the items to be observed at the head of a column. For example:

	1	2	3	4	5				
Black, Sandra									
Brown, John									
Brunson, Joe									

1. Order of quickness in following directions for dividing rectangles into five equal parts.
2. Order of deciding how to divide one side of a rectangular fifth into two equal parts.
3. Order of seeing similarity between Fig. 7 and Fig. 8.

Again this is only a suggested way. Each teacher should discover his own most effective way of observing and recording information which he wants to use concerning the mathematical abilities of his pupils.

To prepare for the actual testing the teacher should cut a rectangle four inches by twenty inches from a sheet of yellow construction paper for each child in the class. After each child has been provided with a rectangle, a ruler, scissors, and pencil, the teacher can ask the pupils to divide the rectangle into five equal parts and label each part as a fraction. By careful observation, the teacher will be able to note which pupils promptly solve the problem either mentally or by finding one-fifth of twenty on paper and then proceed to apply the partial solution of the problem to the task assigned. This observation

on the teacher's part is a very important feature of the test. It is so important that the teacher should usually record the names of the pupils in the order that they solve the initial problem and proceed to the application. He will then have the rank order of the class in respect to their ability to apply previously learned facts to the solution of a new problem.

When the rectangle has been divided into five equal parts and labeled as in Fig. 1, give each of the pupils another rectangle of a different color (green for example) but the same size and ask them to divide it into fifths and label as in Fig. 2. Ask the children to cut the 1st *fifth* off this green rectangle. Then ask them to divide that fifth into two equal parts, as in Fig. 3 and label as indicated: *a*, *b*. Also have them label each fraction part by asking them what fractional part of the piece of paper they are holding is "*a*"; is "*b*"; and record how many recognize at once that each part is $\frac{1}{2}$ of the *square being considered*.

To discover how much they know about another mathematical concept, have the children take the yellow rectangle and divide the first small rectangle into two equal parts and label each part the correct fractional part of the total rectangle as in

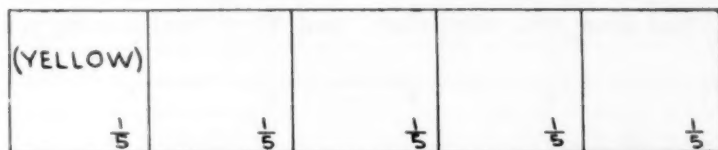


FIG. 1

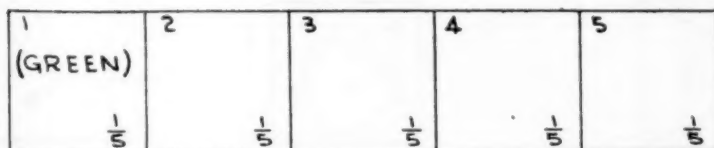


FIG. 2

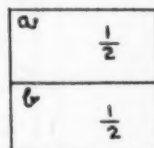


FIG. 3

Fig. 4. This can indicate how many of the pupils have the concept that the same sized piece may have different fractional values when it is taken from different sized figures. Of course this would only be used if the teacher was sure the children had previously had enough experience to deal with this part of the problem.

At this point the teacher could ask the pupils to prove that the part labeled $\frac{1}{10}$ is correct and observe the way they go about it. Again he could record the order in which the pupils reach the correct proof.

At this point the teacher has discovered at least two things about the pupils' knowledge, namely how many of the class could label each part (a and b) with the correct fractional part of the rectangle in Fig. 3, and, second, how many could write what fractional part it was of the rectangle in Fig. 4 section 1. This information will show which pupils have some understanding of the meaning of fractionations and help in grouping the class for further instruction.

Next have each pupil cut the fifth that is numbered (5) in Fig. 2 off the green rectangle. Then, observing carefully the order in which the pupils correctly do as they are asked, have them use their rulers to divide the left side of this fifth into two equal parts. Place a dot exactly at the point that will divide the side into two equal parts. Then draw lines from that

point to the opposite corners of the fifth. See Fig. 5. Ask the pupils to label each fractional part of this rectangle and prove that they are correct. Again the teacher can use a check list to record in order the names of the pupils who can prove that they have labeled the parts correctly. From that data he will be able to determine certain facts about the level of understanding of fractional relations of the individuals in the class. He can easily check his list by walking about observing the children who have cut the figure on the lines which they drew and have placed the cut pieces together on the center triangle as in Fig. 6. Now have each pupil draw these same lines in the yellow rectangle Fig. 1, part (5), and label them the fractional part of that total rectangle. For example see Fig. 7. This part can be used in the upper grades as a somewhat more advanced check for the concept that the fractional part of one unit will have a different fractional value of a larger unit. The difficulty would be somewhat greater than in the previous experience of this kind as can be readily seen.

To continue with the test, ask the pupils to cut the *fifth* labeled (4) from the green rectangle and divide it exactly as they did with the *fifth* labeled (5). Observe again to discover which of the pupils, who may have had trouble the first time, were able to do it quickly and correctly the second time. Have them go to the yellow rectangle

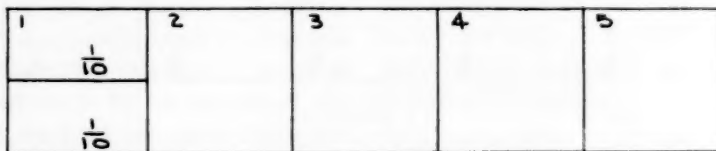


FIG. 4

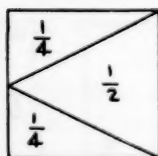


FIG. 5



FIG. 6



Fig. 1, draw the lines in the fourth, *fifth* and write the correct fractional part of the whole rectangle as in Fig. 8.

The teacher can prepare test questions at this point to discover how many pupils know how many of the $\frac{1}{20}$ parts make $\frac{1}{10}$, or what you would do to the $\frac{1}{10}$ part to get $\frac{2}{20}$ of the large rectangle, or others depending upon the maturity and ability of the pupils and the purpose of the teacher.

The next step is to cut off the *fifths* labeled (3) and (4) from the yellow rectangle so that they are together as in Fig. 9. (Note that each task gets a little more difficult). Ask the children to divide the line between the two *fifths* in two equal parts as in Fig. 10 and place a dot.

Draw lines from that dot to the two upper corners. See how many of the class can label the fractional parts of Fig. 10. (Consider the total of Fig. 10 as the unit.)

Take the yellow rectangle and draw

similar lines in the second and third *fifth*. Then ask the children to label each part to find out how many recognize that the parts then become as in Fig. 11.

The teacher then has tested $\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{20}$ in several different relationships. If the children have been able to follow the directions and make the applications the teacher has some evidence about their understanding of fractions that a conventional test would not reveal.

The test can be completed in this way. Have all pieces of paper removed except Fig. 1, which should look like Fig. 12 if it has been developed correctly.

As soon as all pupils have removed every piece of paper from their desks, except the yellow rectangle (Fig. 12) have them cut out the parts and arrange them on a white sheet of construction paper as indicated in Fig. 13.

Observe carefully to find out how quickly the individuals in the class can

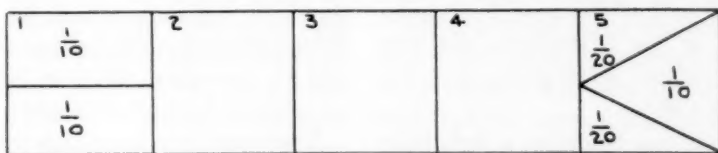


FIG. 7

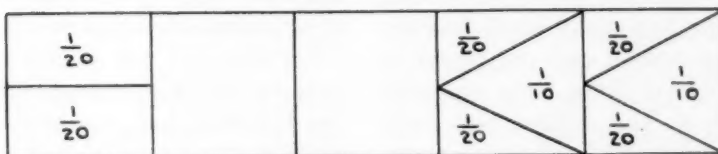


FIG. 8

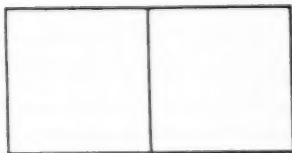


FIG. 9

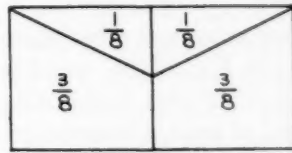


FIG. 10

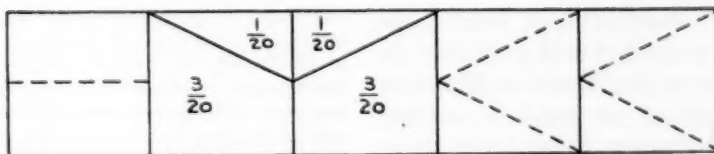


FIG. 11

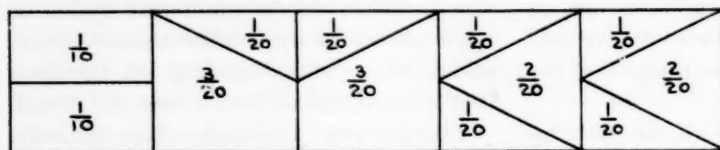


FIG. 12

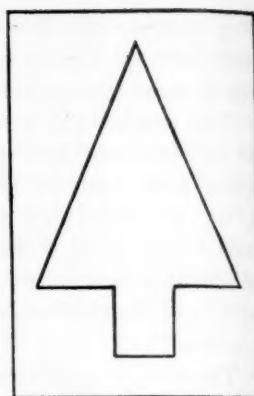


FIG. 13

solve that problem; how many stick to it until it is complete; how many want help; how many have to be given help. Next ask them to turn all the pieces over so that the fractions they have written on the pieces are faced down. Then ask them to remove one-fourth of the tree and prove it. Finally, have the class show all the possible combinations of pieces that could be removed to make one-fourth of the tree: five of the $1/20$ pieces; or one $1/10$ piece and two $1/20$ pieces; or one $1/10$ piece and one $3/20$ piece and so on.

The children can compose a definition of protean at this point. The teacher can ask them to write their definitions so that he can discover how many pupils had the clarity of observation and the power of deduction to lead them to a correctly stated definition. The children can verify their own definitions by consulting a dictionary after the teacher has checked his record sheet.

There are numerous other figures similar to the tree in Fig. 13 that could be formed and many other uses that could be made of this kind of testing-learning problem.

Perhaps some teachers who may try it should be advised that it will take more time than may be anticipated. Several different short periods would probably be better than one or two long ones. However, from fifth grade up the pupils do not lose interest in the problem if they know at the beginning that they are going to make a

protean puzzle, that they will be able to tell what a protean puzzle is when they get it ready to use and if each step is complete and correct for every pupil before the teacher proceeds to the next step.

This procedure in testing has many values for the pupils and for the teacher. It becomes possible for the teacher to detect at each step of the problem which pupils do not understand. He can reteach if he wishes, then proceed to the next step and in most cases the next step is a check on what he has just retaught.

Everyone who tries this way of testing will invent ways to use it to discover aspects of learning which are related to his most important objectives.

Certainly he will soon see that he is learning some very revealing things about the thinking and personal traits of his pupils that were not evident when he used the usual teacher-made or standardized tests of arithmetic ability.

EDITOR'S NOTE: While Mr. Moyer directs his discussion toward testing it is actually a wholesome combination of learning and testing or evaluation. Protean or variable forms or designs seems ideally suited to the concept of a fraction which deals with parts. Mr. Moyer suggests careful observation and questioning of pupils while they are at work in order to ascertain their quickness of perception and depth of understanding. This type of testing is important, it gives insights into pupils' learning which cannot be obtained from ordinary written tests. Other teachers will have similar devices which they use for guiding learning and for evaluation. Let us not be afraid to try new things. Sometimes it is surprising how well they work.

A Modest Proposal

To the Teachers of Developmental Arithmetic for Solving the Many Problems that Beset Their Subject

WALLACE MANHEIMER

Franklin K. Lane High School, Brooklyn, N. Y.

IT IS A MELANCHOLY PROSPECT that the teaching of developmental arithmetic presents to us. Educators could afford to scoff at the attacks upon the program by the least cultured elements of our country. However, insidious doubts spread fast in troubled times; and now even our children are starting to reflect the subtle poison.

Lest you doubt that the teaching of developmental arithmetic is in mortal danger, let me cite a few—only a few—of the events that recently have come to pass. As you will observe, they take place not in an isolated community, but throughout the length and breadth of our great land.

The first episode occurred in a town that we shall call Upper Ramp, in one of our eastern states. In a fifth grade class, a pupil, dividing 4628 by 92, was distinctly heard to mutter, "9 goes into 46 how many times?"

It was only the alertness of the little girl's teacher, a Miss Adelaide Potts, that prevented this from growing into a critical situation. Recognizing the statement as tending toward a virulent and highly infectious type of computationalism, the teacher immediately placed the girl under quarantine in the principal's office. A week of concentrated instruction removed all irrational tendencies in the child, who was returned to class saying, "How many tens of 92's can be subtracted from 462 tens?"

Investigation revealed that the source of the trouble lay with the little girl's grandfather, a former service station attendant, who had used the archaic method in helping her with her homework.

The second illustration comes from the South and can be described as a consequence of gambling with squared material.

Several third grade lads were passing some leisure time in the schoolyard by tossing cardboard arithmetic squares against a wall—winner take all. Their teacher was not alarmed at this natural, boyish occupation, and even was gratified to see that functional life adjustment training was taking place in their exchanging 10 squares for one strip and ten strips for one large square.

However, the class bell suddenly rang, and one of the boys remarked, "Let's see, I owe each of you four fellows sixty chips, and that makes two hundred and forty!"

Horrified at these words from a pupil who was not yet permitted to use symbols for numbers, the teacher immediately challenged him to describe what he meant. The boy, of course, was completely unable to explain the statement, and evidently had not even grasped the fact that he had used the distributive law on a second degree polynomial with base 10.

Interestingly enough, the root of the infection was traced this time to the school itself. The third grade reader used numbered pages, actually going as far as 256; and the boy had, as he described it, "played" with them. The book was immediately recalled and a new edition issued with page numbers indicated by pictures of squared material. Nevertheless, the spore of computationalism had again fastened upon a young and impressionable mind.

The third illustration is by now widely

known. I refer, of course, to the "Anti-Partitionist" movement of the Northwest that has recently become such a national educational scandal.

To the best of our knowledge the story began in the classroom of Miss Wilhelmina Scream, one of the most progressive and widely respected champions of developmental arithmetic west of the Mississippi. Miss Scream was teaching her fourth grade class the difference between partition and quotient in division, using the syllabus approved example of an equal distribution of 35 books. She took pains to point out that if we ask how many books can be given to each of 7 classes we are using *partition*, whereas if we ask how many classes can receive 7 books apiece, we are solving a problem in *quotition*.

At this a pupil arose and delivered this obviously carefully rehearsed harangue:

Pupil: But isn't the answer 5 in each example?

Miss S: Yes, but we are really finding a different thing each time.

Pupil: And didn't you show us last week that 5 times 7 gives the same answer as 7 times 5?

Miss S: Of course. The commutative postulate of multiplication. . . .

Pupil: Then why can't we just say that 35 divided by 7 is always 5 and worry about what the answer means after we get it?

There was a chorus of approbation from the class, and despite all of Miss Scream's efforts the pupils refused to make any study of the difference between partition and quotient.

The problem became so serious that the cooperation of the Parents' Association was sought. However, to the astonishment of the faculty, the parents immediately sided with their children. The Executive Board of the Parents' Association finally issued this statement:

"We parents have gotten along very well without partition and quotient. We simply divide. In our opinion, this also is good enough for our children.

"Nevertheless, we approach the problem in a spirit of compromise. You have taught our children that the answer to a problem in division is called a quotient. Very well, let there be quotient. But NO partition!"

The Anti-Partitionist movement, as it came to be called, expanded rapidly. A self appointed wit coined the slogan, "Fifty-Four-Fortieths or Fight!," and the battle was on. At the present time the movement, already characterized as a national educational disgrace, has spread over more than four of our Northwestern states.

It would be wearisome to multiply instances of the widespread outgrowth of this irrational and mechanistic tendency. Suffice it to say that it is rampant, that it is persistent, and that it is thoroughly malignant.

Obviously any step to rescue the developmental arithmetic program will be welcomed by true educators; and the one that is herewith presented has the inevitability, the efficacy, and above all the simplicity that is the mark of a genuine educational advance. What is the proposal? It is, simply and baldly: RETURN TO THE ROMAN NUMBER SYSTEM!

Let me say immediately that I know there will be conservative scoffers who think the Arabic system is ordained by Divine Right, and who will point mockingly at the use of Roman numbers on cornerstones or in the copyright dates of paper backed reprints. To these I say only that the entire history of education has been marked by constant experiment and innovation. May Heaven preserve us from a system that has become so rigid and ironbound that it precludes the possibility of change and progress!

Let us, then, consider the advantages of the Roman number system. First and foremost, it will quickly stamp out all

tendencies to work mechanically. Understanding will reign supreme! For instance, do you remember how our pupils constantly forget the principle of exchange that underlies subtraction? How they use mechanical processes in its place? In the Roman number system this *cannot* happen. The student who subtracts XVII from CVI *must* exchange the C for ten X's. Thus heedless mechanism will be stamped out.

There may be some who will complain that the proposed methods will be slow compared with the Arabic system. To these critics I can only point out that teachers of developmental arithmetic have never claimed speed as one of their objectives. The pupils we train, it is admitted, may be slower, but they will *understand* what they are doing. Surely comprehension is to be preferred to blind, unreasoning speed, and comprehension is what the Roman number system will give us.

It is easy to find other, concomitant advantages. No longer will there be rejection of squared material, flannel boards and other realia that have made arithmetic meaningful and functional in life adjustment situations. Habits of introspection and criticism will take the place of mechanical motions. And finally—if the author may be permitted a personal reflection—the Roman numbers possess a flexibility, a character, yes, a charm, that will forever be denied to the machinelike Arabic system.

It is hoped that educators will soon rally about the Roman number system at this late date as the sole reliable method of rescuing developmental arithmetic from the quicksands that threaten it on every side. Act now! Save our children before it is too late!

EDITOR'S NOTE: Mr. Manheimer has an interesting proposal. Are some schools and teachers spending too much time on the developmental phases of arithmetic and not producing results upon which the typical high school program can be built? A good agriculturist knows how and when to prepare the seedbed, when to plant, how to nurture growth, and when and how to

harvest. Each stage is important and requires a good deal of discernment and "know-how" and so it is with the teacher of our young. Each stage is important and it is the skilled teacher who knows when and how to move from one to another.

BOOK REVIEWS

Making Sure of Arithmetic, Grades 3, 4, 5, 6, 7, and 8. *Teacher's Edition*, *Making Sure of Arithmetic*, Grades 3, 4, 5, 6, 7, and 8, Morton, Robert L., Merle Gray, Elizabeth Springstun, and William L. Schaaf. Silver Burdett Company, 1946, 1952, and 1955.

Making Sure of Arithmetic was first published in 1946, followed by significant revision in 1952. The current edition is the result of certain revisions in 1955.

The first major change is the addition of new material to stimulate the more able pupil. This is accomplished by inclusion of a new section entitled, "How Far Can You Go in Arithmetic," at the end of each book. As an example, Chapter Two in the sixth grade book contains a review of the preceding work in fractions followed by an extension of those principles. One topic is addition and subtraction of mixed numbers containing like denominators. The pupil shows each step in doing any given example. However, the more able pupil can be referred to page 348, where examples of the same type are done mentally. After reading several illustrative examples, the pupil may do a set in that manner.

In the new *Teacher's Edition* each page of the text is reproduced. The answers to any examples and problems on that page are given, as well as suggestions to the teacher. Each *Teacher's Edition* reviews the basic philosophies of arithmetic teaching behind the series and gives an overview of the entire program.

This series departs from the teaching of arithmetic as a fixed sequence of topics with complete mastery and exploration of one topic before beginning another. A desirable, logical order is followed. However, exploration of many topics is begun early and then expanded in each succeeding grade. The concepts of multiplication and division are begun in Grade Three. Concepts of geometry are begun as early as Grade Four.

The authors feel that reteaching all basic concepts at each level is essential. Therefore, at no grade level is any topic dropped. Instead each basic concept is reviewed at a higher level. It is essential for each teacher to be familiar with the whole program.

These books should appeal to the pupil. The liberal use of color makes the cover and pages of each book attractive. The pages seem easy to read and well illustrated. The content seems well chosen. It relates to a pupil's everyday experiences and is up-to-date. The content will also tie in with teaching in other areas. These textbooks contain a sufficient amount of material for practice in computation and problem solving. As is customary, there are frequent review sections. There is ample work on estimation of answers.

One change the authors might consider concerns making out bank deposit slips in Grade Seven. In listing checks, no information other than the amount of each check is listed. Usually, checks are listed by their transit number or name and location or some combination of these.

Testing programs and workbooks are available for this series.

ROBERT VAN DAM

The Scribner Arithmetics, Books 3, 4, 5, 6, by Richard Madden, Leslie S. Beatty, William A. Gager. Book 7 by William A. Gager, Beulah Echols, Carl N. Shuster, Richard Madden, Franklin W. Kokomoor. New York, Charles Scribner's Sons, 1955.

These books offer a carefully developed course in an attractive package. The inexperienced teacher will find them stimulating and helpful. The experienced teacher will value them for the care with which the work is presented and for its variety. We have come to rely on expository materials for teachers as an important part of a series of textbooks in Arithmetic. In the case of the *Scribner Arithmetics*, the "Teacher's Guides" for each grade give teaching suggestions, additional work for pupils, and ideas for enrichment materials.

The authors have consistently kept to the policy of basing each new item on an experience which the pupils have had or which they can easily picture themselves as having. Concrete materials are used in studying this item and in many cases, these materials have been made by the pupils. The pupils are then helped to express this specific discovery in words and to represent it in abstract form.

The authors advocate the use of an "Answer Strip" for drill in number facts, and to identify the facts that need further study. The answer strip is a narrow piece of ruled paper. The lines are numbered and the pupil writes the answers as the teacher gives the questions. Arranged side by side, these strips facilitate the quick spotting of individual and class weaknesses. The questions are heard not read. This makes for greater control over the time element. Mary can't dawdle over a certain item and fail to try those that follow. The teacher is not bound by the tests or drills printed in the text but may make his own list suited to the class on this particular day.

Each unit of work has its test for pupil mastery and, beginning with Grade 4, Growth Tests and Diagnostic Tests are provided to measure pupil progress and to indicate items that require study.

The diagrams are excellent. In Book 4, for instance, a bar graph shows the comparative sizes of a thousand, a hundred, ten, and a unit. In Book 5, the number 1000 is shown effectively by 100 close-set rows of 10 dots each.

In view of the care with which these texts have been written, it is surprising to see an occasional inexact statement as "You can carry in multiplication in the same way you carry in addition." It is unusual, too, to combine the division algorism and the equality sign as has been done in the statement (Book 6, p. 147)

$$2\overline{)4} = 3 \times 2 \overline{)3 \times 4} = 6\overline{)12}$$

In Book 4, the authors introduce the names of the numbers in the four fundamental operations. This is logical but it seems forced. It may be questioned whether the word *addend* need be learned at all. Is it necessary at this point to learn that the number you are subtracting is the *subtrahend*?

The Scribner Arithmetics are noteworthy for the early introduction of certain topics. In Book 4, for example, the pupils are given experience in estimating answers, and in selecting answers that are reasonable. They read distances from a map, round these distances to the nearest 10 miles, read bar graphs and find averages. These topics are developed in the later volumes, but their early introduction is important.

The authors put considerable stress on the study of relationships. In Book 4, the pupils investigate what happens to the size of a part of a thing as the number of equal parts increases, and what happens to the quotient if the dividend stays the same but the divisor increases, or if the dividend changes while the divisor stays the same. In Book 5, this idea is extended to the multiplicand-multiplier-product relationship. In Book 6, the missing number in a statement is represented by N as $N+15=20$ and $N \times 8=32$. *The Scribner Arithmetics* expect the pupil to use his brains.

VERA SANFORD

What Is Zero?

When the temperature is zero degrees this does not mean *no* temperature? Zero sandpaper does not mean *no* sandpaper. Various items of merchandise have sizes of zero. What does this mean? Zero as a beginning point on a scale such as is illustrated in latitude and longitude has many modern uses. Zero as a number in our system of notation is used to indicate none or nothing as when the score is 6 to 0. But zero is also used in a slightly different sense when a number like 605 uses a zero to fill the "place" which otherwise would be vacant and would lead to confusion if we had no convenient way to identify the proper "places" of the 6 and the 5 in the number system.

A Coin Trick

Suppose you have nine coins identical in size and appearance but one is counterfeit and heavier than the other eight. How can you with a simple balance scale locate the counterfeit coin in just two balance tests? For your first step place any three on one side of the balance and any other three on the other side. Now complete the test and find the counterfeit coin.

OFTEN THE NUMBERS ARE MISSING

If you stop to consider the way in which functional mathematics is presented to the average person it is easily apparent that many of the situations do not present numbers and frequently these situations appear pictorially and sometimes they are oral. If we want functional competence, we must provide for several different avenues of learning and testing. Following are some exercises that are a little different from the typical test items.

1. George has these coins:



How much more money does he need in order to buy a knife costing 40 cents?

2. In guessing the number of beans in a jar, the nearest number to the correct answer of 913 wins a prize. Which of the numbers below is nearest to 913?

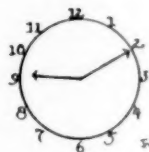
(A) 899 (B) 930 (C) 500 (D) 1000 (E) 931



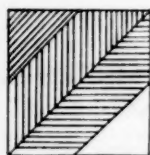
3. If we know how many quarts of berries each of five boys has picked, how can we find the number of quarts picked by all the boys?

(A) add (B) subtract (C) multiply (D) divide

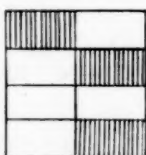
4. How much later is the time shown on the clock at the right than the time shown on the clock at the left?



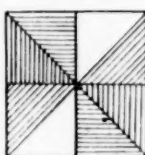
5. Which of the squares below has had three-fourths of its surface shaded?



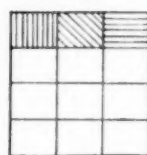
(A)



(B)



(C)



(D)

6. Which of the statements below is not a correct way of writing the value of one-half cent?

(A) $\frac{1}{2}\text{¢}$ (B) \$.005 (C) \$.00 $\frac{1}{2}$ (D) .5¢ (E) \$.0 $\frac{1}{2}$

7. How long is the stub of a pencil pictured below?



(A) 3 in. (B) 3 $\frac{1}{2}$ in. (C) 4 in. (D) 4 $\frac{1}{2}$ in.

8. Which of the following rates of profit cannot be true?

(A) 5% of the cost (C) 200% of cost
(B) 150% of selling price (D) 50% of selling price

How Are Your Nines?

ROBERT C. BANE
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HAVE YOU EVER WISHED for an abundance of computational material whose accuracy could be verified when teaching the fundamental operations of addition, subtraction, multiplication and division? All of us have, no doubt, yet I tremble to think how many times I have overlooked the tremendous possibilities inherent in the casting out of 9's before I ever made any extended use of this principle for checking the accuracy of arithmetic computations.

The system of casting out 9's which is here explained is widely used in Europe, and in most European countries it is used as a base upon which the whole structure of computational mathematics is erected.

For those who have forgotten, or who are unfamiliar with the meaning of the expression, an explanation of the mechanics of casting out 9's may be in order.

This operation is based upon the observation that if the digits of any number are added together, and the digits of any number resulting from this operation are given the same treatment until the final total of digits is less than nine, the result will be the same as the remainder obtained when the original number is divided by nine. The only catch is that when the final total of digits is 9, it is necessary to write zero instead of nine.

Let us apply this principle first to a number of five digits, in order to grasp the basic mechanics of casting out 9's. In the number 83,524, if the digits are added, the total is 22. If the digits of 22 are added, the result is 4. You will see that this is the same as the *remainder* obtained when 83,524 is divided by 9. (From here on in, I am afraid you are going to need pencil and paper.) This operation is usually indicated thus: $83,524 \rightarrow 4$ and is read, "83524 yields 4."

Now let us apply the technique of casting out 9's to checking a small addition problem.

$$8356 \rightarrow 5$$

$$1741 \rightarrow 4$$

$$2653 \rightarrow 7$$

$$4861 \rightarrow 1$$

$$17611 \rightarrow 7$$

If we cast out 9 in the sum, 17,611, the result is 7. If we add the "yields" obtained by casting out 9 from each of the individual addends, and continue of course until the total is less than 9, the result will also be 7. Above, the total of the "yields," not including the 7 we get from the 17,611, is 16, which itself yields 7.

Now we can attempt a longer and more difficult problem in addition.

$$7643 \rightarrow 2$$

$$13582 \rightarrow 0$$

$$7691 \rightarrow 5$$

$$8432 \rightarrow 8$$

$$7802 \rightarrow 8$$

$$9641 \rightarrow 2$$

$$44791 \rightarrow 7$$

The total of the "yields" from the individual addends is 25, which of course yields 7. This is the same as the result obtained by casting out 9's in 44,791.

As your skill in casting out 9's develops, it will be possible to speed up the work by alerting your powers of observation. Consider the problem just finished. When we look at the number 7643, we may think as follows:

$$7643 \rightarrow 20 \rightarrow 2$$

It is possible to shorten this work by recognizing that in 6743, 6 and 3 make

nine, and need not be considered in reaching the final result. The work would then assume this form:

$$7643 \rightarrow 11 \rightarrow 2$$

In the same way, $8432 \rightarrow 17 \rightarrow 8$. But strike off 432, which total 9, thus: $8432 \rightarrow 8$.

Now let us apply our system to a subtraction problem:

$$\begin{array}{r} 83742 \rightarrow 6 \\ 65942 \rightarrow 0 \\ \hline 17799 \rightarrow 6 \end{array}$$

If we cast out 9 from the difference, the result will be 6, which is the same as we get when we subtract the zero yielded by the subtrahend from the 6 yielded by the minuend.

Do you want to try another one?

$$\begin{array}{r} 82965 \rightarrow 3 \\ 40513 \rightarrow 4 \\ \hline 42452 \rightarrow 8 \end{array}$$

Stymied? No, it is not possible to subtract 4 from 3 and get 8, but it may occur to the logical mind that if 9 can be cast out, it can also be inserted, and that the problem will then appear in this manner:

$$\begin{array}{r} 82965 \rightarrow 3 \quad 9 = 12 \\ 40513 \rightarrow 4 \quad = 4 \\ \hline 42452 \rightarrow 17 \rightarrow 8 \end{array}$$

In other words, if the result obtained by casting out 9 from the subtrahend is too large to subtract from the number so obtained from the minuend, add 9 to the "yield" obtained from the minuend and proceed as before.

Now we are fortunate indeed, for any computations involving only addition and subtraction may be checked anywhere in the world, even under the trees of the forest, and infallibly verified without adding machine, calculator or tables.

The system may also be applied to problems in multiplication. First we will solve and check an easy problem.

$$\begin{array}{r} 832 \quad 4 \\ 741 \rightarrow 3 \\ \hline 832 \quad 12 \rightarrow 3 \\ 3328 \\ 5824 \\ \hline 616512 \end{array}$$

The result obtained by casting out 9's from the multiplicand is 4. The result obtained by casting out 9's from the multiplier is 3. Their product is 12. Cast out 9's from 12 and you have 3. This is the same as the result obtained by casting out 9's from the product of the multiplicand and multiplier, thus: $616512 \rightarrow 21 \rightarrow 3$.

Would you like to try another one? Here goes:

$$\begin{array}{r} 373 \rightarrow 4 \\ 241 \rightarrow 7 \\ \hline 373 \quad 28 \rightarrow 1 \\ 1492 \\ 746 \\ \hline 89893 \rightarrow 10 \rightarrow 1 \end{array}$$

Now let's see if the process can be used to check the work of division.

$$\begin{array}{r} 607 \\ 12 \overline{) 7284} \\ 72 \\ \hline 84 \\ 84 \\ \hline \end{array}$$

Cast 9 out of the divisor and the result is 3. Cast 9 out of the quotient and the result is 4. The product of 4 and 3 is 12, which yields 3 when casting out 9's thus: $21 \rightarrow 3$. Likewise, $7284 \rightarrow 21 \rightarrow 3$, or $7284 \rightarrow 12 \rightarrow 3$.

Now we have checked all four of the fundamental operations, but have not yet considered division in which a remainder is involved. That means a division in which the divisor is not contained an integral number of times in the dividend. We may reason by analogy, a process which often leads to error, but is nevertheless helpful

in this case. To check by arithmetical methods a division problem in which a remainder is involved, we multiply the divisor and the quotient, after which the remainder is added to the product so obtained.

To check division by casting out 9's, we cast 9 out of the divisor and quotient, multiply the "yields," and add to the product the number obtained by casting 9 out of the remainder. The result will be the same as that obtained by casting 9's out of the dividend. For example, when 7278 is divided by 83, the quotient is 87 and the remainder is 57.

$$\begin{array}{r} 83 \rightarrow 2 \\ 87 \rightarrow 6 \\ \hline 12 \rightarrow 3 \\ 57 \rightarrow 3 \\ \hline 6 \end{array}$$

The "yield" from 7278 is 6.

These processes, which seem so clumsy while you are learning them, may be applied with unbelievable speed and accuracy. There is no computation, involving only integers, which can not be checked by casting out 9's. When the matching numbers have been reached, the solution is accurate. If results are slow in coming, it is often wise to go back and

repeat the work which is being checked, as any error in the original computations is naturally reflected in a subsequent unsuccessful attempt to reach matching numbers by casting out 9's.

It is perhaps best to try to develop your skills by checking addition problems until you can handle them with facility, and then to attempt the more involved work of checking subtraction, multiplication and division.

EDITOR'S NOTE: Mr. Bane is a firm believer in the "check of nines." But this check is not infallible. Consider the following:

365 - 5	3684 - 3
722 - 2	$\times 203 - 5$
484 - 7	<hr/>
391 - 4	11052 15 - 6
784 - 1	73680
<hr/>	<hr/>
2656 - 1	84732 - 6

A good check should (a) discover certainly if there is an error, (b) be easily applied, (c) employ worthwhile processes in its application, and (d) locate the precise position of the error. On these standards the "check of nines" rates only fair. Recently a group of experienced and well informed teachers were asked if they used the "check of nines" and about half of them said they did but only in certain circumstances. Pupils in grades six, seven, and eight seem to like it very much. It has some real "process value." in that it uses combining digits and as Mr. Bane points out, there are various stages of maturity in its operation and use.

Inconsistencies in the Teaching of Arithmetic in the Elementary Grades

LESLIE A. DWIGHT

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IN ORDER TO INTRODUCE the subject matter in this article the following true-false test is given for the reader. It is suggested the reader take this test and attempt to rationalize each answer in lan-

guage suitable for elementary grade pupils. Preserve your test answers and compare them and your discussions of the items with those of the author which will be printed in the next issue.

Test Items

1. _____ $5 + 2 \times 5 = 35$.
2. _____ $0 \text{ times } 5 = 0$.
3. _____ A whole number may be multiplied by ten by adding a zero to the number.
4. _____ $2 \frac{1}{3} = \frac{3 \times 2 + 1}{3} = \frac{7}{3}$
5. _____ A two-inch square contains two square inches.
6. _____ Five times the sum of seven and four equals the sum of five times seven and four.
7. _____ John has eight dollars and Henry has four dollars. Compare the value of John's money to the value of Henry's money.
Answer: *It is two times greater.*
8. _____ Addition is a quick way of counting.
9. _____ When subtracting take the smaller number from the larger number.
10. _____ Multiply means to increase.
11. _____ Multiplication is a short way of adding.
12. _____ $2 \text{ feet} \times 3 \text{ feet} = 6 \text{ square feet}$.
13. _____ The symbol 9 has a larger value than the symbol 1.
14. _____ $5 \text{ feet} \times 6 = 30 \text{ feet}$ is read as "five feet times six equals thirty feet."
15. _____ A number may be multiplied by ten by annexing a zero on the right.
16. _____ Mary bought 5 80 ¢
pounds of steak $\times 5 \text{ lbs.}$
that cost 80 cents —
a pound. Find 400 ¢
the total cost of
the steak (solution at right).
17. _____ $5 \times .25 = \$1.25$
18. _____ Increasing the divisor and dividend by same number does not change the value of the quotient.
19. _____ Division is a short way of subtracting.
20. _____ $8 \text{ square feet} \div 2 \text{ feet} = 4 \text{ feet}$.
21. _____ If 12 dollars is divided into two groups each group is called one-half.
22. _____ $8 \div 0 = 0$.
23. _____ $18 \div 2 \times 5 = 45$.
24. _____ $\frac{3}{8}$ divided into $\frac{1}{2} = \frac{3}{2} \times \frac{2}{1} = \frac{3}{1} = 3$
25. _____ $5 \times .25¢ = \$1.25$.
26. _____ In the number $2\frac{1}{3}$ the fraction $\frac{1}{3}$ means $\frac{1}{3}$ of 2.
27. _____ 3 will divide into 21 an even number of times.
28. _____ $.0\frac{1}{3}$ means $\frac{1}{3}$.
29. _____ $.0\frac{1}{3}$ means $\frac{1}{3}$ of $\frac{1}{100}$ or $\frac{1}{300}$ hundredth.
30. _____ In the subtraction exercise at the right 7 tens cannot be taken from 2 tens so we take (or borrow) one hundred ones from the three hundreds and change it to tens.

EDITOR'S NOTE: In this issue we are printing only Part I of Dr. Dwight's article. In the next issue his rational explanation of each item will be given. A number of the items depend upon an understanding of rules of operation while others call for clarity of expression. It is not solely in the correctness of the answers that the value of the test lies but more in the explanation. Dr. Dwight has suggested that we rationalize each answer in language suitable for elementary grade pupils. That is a challenge.